Origami with conformal and helical symmetry

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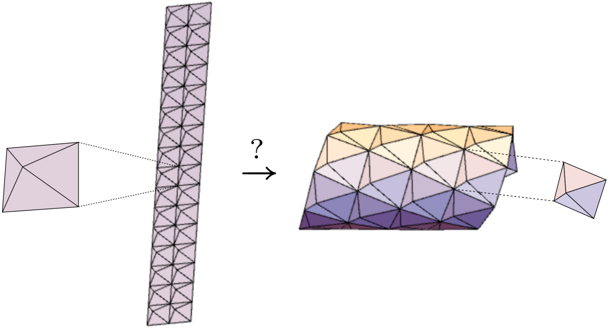
Helical miura-ori (F. Feng, P. Plucinsky, R.D James (2019)).

Background: we want to fold this tube from the **flat** crease pattern (left) - flat things are easier to fabricate, store, etc.

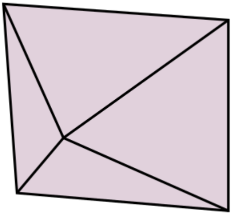


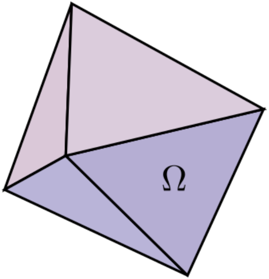
Question

Mathematically, what is the simplest way to rigidly deform the crease pattern into the folded tube?

Helical miura-ori (F. Feng, P. Plucinsky, R.D James (2019)).

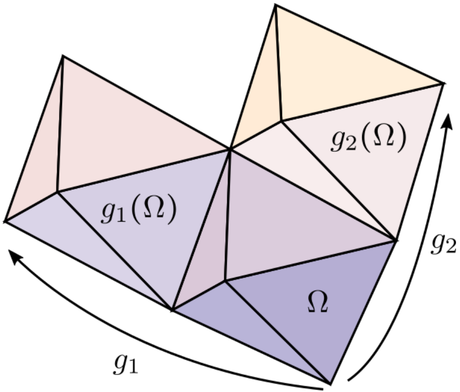
The idea: to design origami using **groups**.





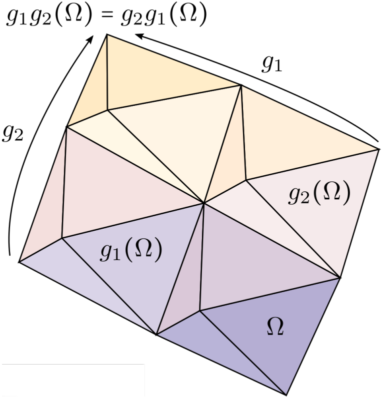
Take a flat unit cell (parallelogram on the left), and fold it slightly (folded tile Ω on the right).

Helical miura-ori (F. Feng, P. Plucinsky, R.D James (2019)).



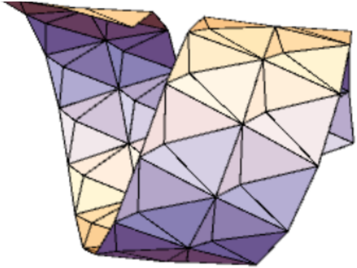
Find a pair of **commuting screw isometries** g_1, g_2 mapping opposite edges.

Helical miura-ori (F. Feng, P. Plucinsky, R.D James (2019)).



The commutativity assumption ensures that the fourth cell fits - **local compatibility**.

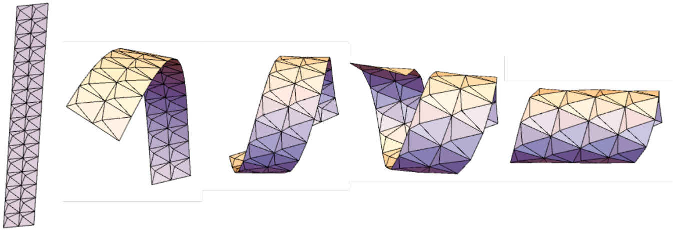
Helical miura-ori (F. Feng, P. Plucinsky, R.D James (2019)).



Applying some powers of g_1, g_2 gives a **locally compatible structure**.

Helical miura-ori (F. Feng, P. Plucinsky, R.D James (2019)).

There is an implicit dependence on the **folding angle** ω of the unit cell. Increasing ω, the resulting structure will close.



The closed structure corresponds to a relation:

$$g_1^p g_2^q = 1, \quad \text{for some } p, q \in \mathbb{Z}.$$

- the **discreteness condition**.

Helical groups

We use the notation $(A|b)$ for the map $x \mapsto Ax + b$. Explicitly, these groups are generated by g_1, g_2 of the form

$$g_i = (R_{\theta_i}|(I - R_{\theta_i})z + \tau_i e), \quad i = 1, 2,$$

where the R_{θ_i} have axis e, and the discreteness condition reduces to:

$$p\theta_1 + q\theta_2 = 2\pi,$$
$$p\tau_1 + q\tau_2 = 0,$$

for some $p, q \in \mathbb{Z}$.

Goal

Can we generalise this approach to groups of non-isometries?

Conformal euclidean groups

We look at the **conformal Euclidean groups**: elements are transformations of $\mathbb{R}^N$ of the form

$$(\lambda Q|c), \quad \lambda \in \mathbb{R},\ Q \in O(N),\ c \in \mathbb{R}^N.$$

and containing some element with $\lambda \neq 1$.

Question

Which such groups G satisfy the **semi-discreteness condition**

$$\inf_{\substack{x_1,x_2 \in (Gx)' \\ x_1 \neq x_2}} |x_1 - x_2| > 0,$$

for all $x \in \mathbb{R}^N$? i.e., accumulation points are spaced.

Conformal euclidean groups

Theorem

The conformal Euclidean groups satisfying the semi-discreteness condition are precisely those of the form

$$\{\tau g_0^k h \tau^{-1} : k \in \mathbb{Z}, h \in H\}$$

where:

- $g_0 = (\lambda Q|0)$ *for any* $\lambda \neq 1$, $Q \in O(N)$;
- $H \subseteq O(N)$ *is a point group;*
- $\tau = (I|s)$ *is a translation by* $s \in \mathbb{R}^N$.

Conformal euclidean groups

For applications, we want to consider groups acting on $\mathbb{R}^3$, generated by a pair of commuting generators g_1, g_2. By taking H to be an abelian point group, these are:

1. Accumulating rays.
2. Accumulating lines by reflection.
3. Accumulating lines by mirroring.
4. Accumulating pyramids.
5. Accumulating saddles.
6. **Spiral groups**.

Cases 1-5 are a little "degenerate" (they arise because groups of order ≤ 4 are abelian).
The most interesting is 6 - the **spiral groups**.

Spiral groups

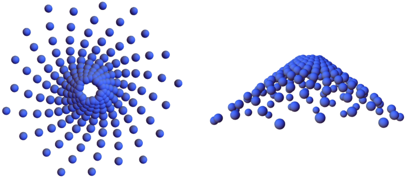
The spiral groups are generated by a commuting pair g_1, g_2, given by:

$$g_i = (\lambda_i R_{\theta_i} | (I - \lambda_i R_{\theta_i}) s), \quad i = 1, 2,$$

where the R_{θ_i} are coaxial rotations, satisfying the semi-discreteness condition:

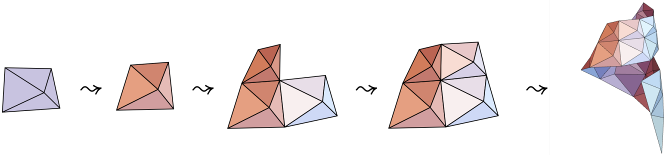
$$\begin{aligned} p\theta_1 + q\theta_2 &= 2\pi, \\ p\log(\lambda_1) + q\log(\lambda_2) &= 0, \end{aligned}$$

for some $p, q \in \mathbb{Z}$.

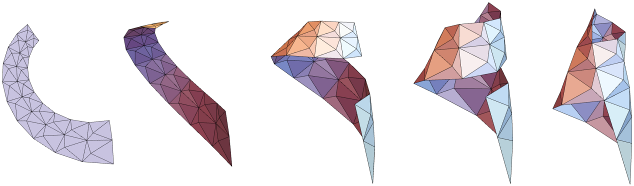


Conformal miura-ori

Using this abelian group generated by g_1, g_2, the same approach works:

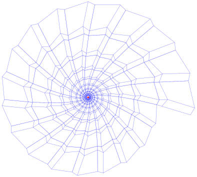
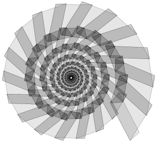
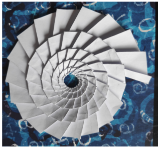
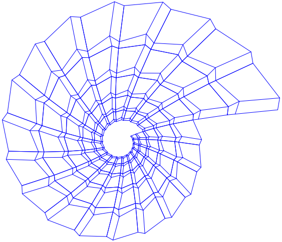
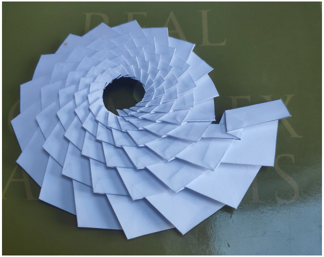


Just as before, increase the folding angle until the semi-discreteness conditions are satisfied:



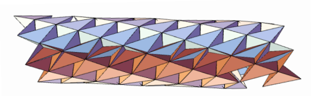
Another use of conformal groups

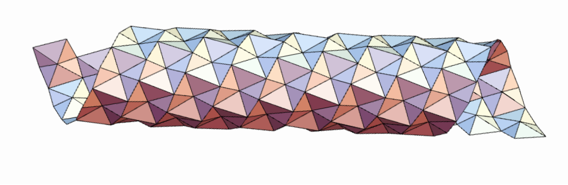
Another application: The 2-d groups can be used to make flat-foldable origami (interesting to enthusiasts).
The construction is similar to the quad-mesh origami (but unfortunately, not rigidly foldable).

Waterbomb origami and inverse approach

Suppose that we want to generalise the construction of waterbomb origami: useful because they exhibit **bistability**.





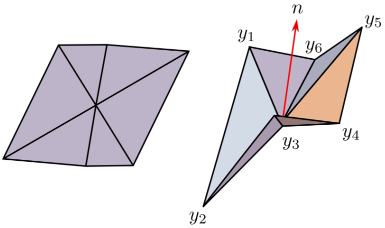
This gives applications, e.g. in medical stents (P. Velvaluri, A. Soor, P. Plucinsky, R. D. James, R. Lima de Miranda, E. Quandt "Origami-inspired thin-film shape memory alloy devices" (2021)).

Problem

The forwards approach is intractable (many DOF).

Waterbomb origami and inverse approach

The unit cell is a "generalised waterbomb cell", with 180° symmetry about axis n, folded symmetrically.



We want compatibility under an abelian helical group generated by g_1, g_2, which amounts to:

$$
\begin{aligned}
g_1(y_2) &= y_6, \quad g_1(y_3) &= y_5 \\
g_2(y_3) &= y_1, \quad g_2(y_4) &= y_6
\end{aligned}
$$

(Commutativity ensures the last pair of edges are compatible under $g_2^{-1}g_1$).

Waterbomb origami and inverse approach

Recall that

$$g_i = (R_{\theta_i} | (I - R_{\theta_i}) z + \tau_i e) \quad i = 1, 2.$$

If we make the (mild) assumption that $z \| n$, then the local compatibility conditions imply that:

$$|P_e(y_1 - z)| = |P_e(y_2 - z)| = \cdots = |P_e(y_6 - z)|.$$

i.e. the six vertices **all lie on a cylinder**.

Idea

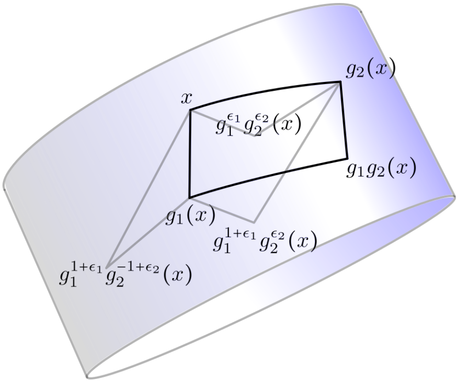
Inverse approach: To make a **closed tube**, choose 6 vertices on the cylinder, compatibly with a **discrete** helical group, and try to "unfold" the resulting structure.

Waterbomb origami and inverse approach

To choose the vertices, we **continuously extend** the generators g_1, g_2 of a discrete helical group. For $t \in \mathbb{R}$, define:

$$g_i^t := (R_{t\theta_i} | (I - R_{t\theta_i})z + t\tau_i e), \quad i = 1, 2,$$

then, choosing $\epsilon_1, \epsilon_2 \in \mathbb{R}$ appropriately, prescribe vertices as:



Waterbomb origami and inverse approach

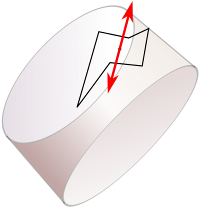
The hexagon tesselates around the cylinder. It has 180° symmetry about an axis n. Choose the central vertex y_0 along the axis, i.e.

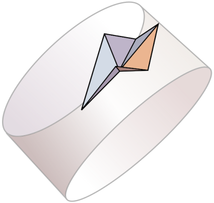
$$y_0 = \frac{1}{2}(y_1 + y_4) + \mu n$$

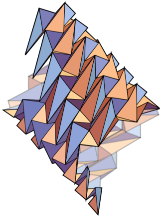
for some $\mu \in \mathbb{R}$, such that:

$$\angle y_6 y_0 y_5 + \angle y_5 y_0 y_4 + \angle y_4 y_0 y_3 = \pi.$$

This ensures that the cell can be unfolded. Easy numerically: the LHS is a simple function of μ.

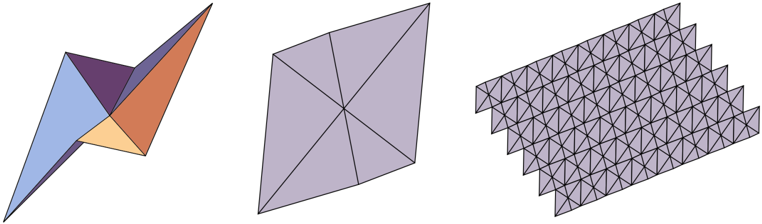






Waterbomb origami and inverse approach

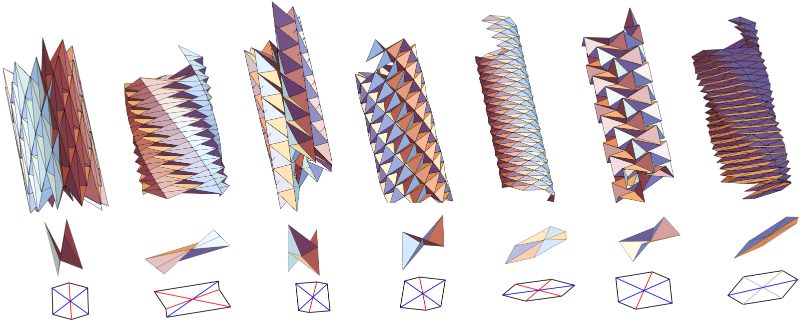
The 180° rotational symmetry is preserved under unfolding. Any hexagon with 180° rotational symmetry can be tessellated: so compatibility in the flat state is automatic.



$\rightsquigarrow$ we get a flat tessellation which folds to give the closed tube we designed.

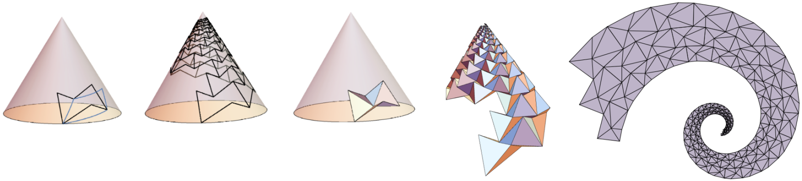
Waterbomb origami and inverse approach

Choosing different parameters leads to many interesting tubes.



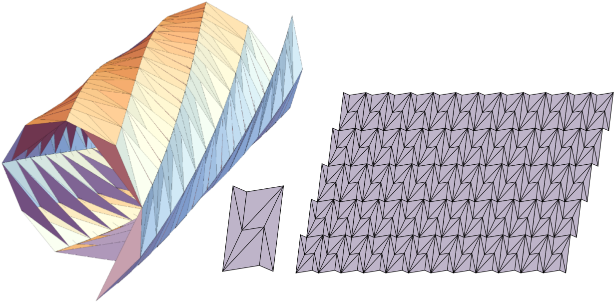
Conformal origami and inverse approach

A similar approach also works for the conformal Euclidean groups: here we choose points on a **cone** rather than a cylinder.



What could you do with these groups?

No reason to restrict to “classical” (e.g. miura-ori, or waterbomb as I have here), for instance: a strange unit cell with two degree-5 vertices.



Thanks!

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