Topics in derived analytic geometry

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To Prof. Charles J.K. Batty on the occasion of his 73^{rd} birthday

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Abstract

We investigate the six operations in the context of relative algebraic geometry. As an application, we obtain a theory of derived rigid geometry, which is shown to have applications to the theory of $\widehat{\mathcal{D}}$-modules of Ardakov and Wadsley.

Statement of Authorship

This thesis contains no material that has already been accepted, or is concurrently being submitted, for any degree, diploma, certificate or other qualification at the University of Oxford or elsewhere. To the best of my knowledge and belief, the work contained in this thesis is original and my own, unless indicated otherwise. Much the the material of Chapters 2 and 3 already appeared in, or is very similar to, my pre-print [Soo24].

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Contents

Chapter 1

Introduction

1.1 Philosophical discussion

The purpose of this thesis is to illustrate the following: that (derived) analytic and algebraic geometry, two subjects one might have thought to be disjoint, can be handled on an equal footing. That is, they are both instances of *relative algebraic geometry.* My hope is that, by the end of this thesis, the reader will appreciate the usefulness of this perspective, which has been introduced by Bambozzi, Ben-Bassat, Kelly, and Kremnizer, and which also appears in the work of Clausen–Scholze [BBK17, BBB16, CS19b]. The following discussion is merely my attempt to explain their insights.

In algebra, we study rings and modules. To rephrase this in a slightly pretentious way, we are studying monoids in the monoidal category Ab of abelian groups, and modules over them. Therefore, idea of relative algebraic geometry (following Toën–Vezzosi [TV08], Toën–Vaquié [TV09] and Deligne [Del90]) is to replace Ab by some other symmetric monoidal category $\mathscr{V}$.

With a judicious choice of such $\mathscr{V}$, we will be doing "analysis". A naïve guess would have one taking $\mathscr{V}$ to be some kind of category of locally-convex vector spaces. It is possible that the development of locally-convex vector spaces was something of a historical wrong turn, because of their bad algebraic and homological properties. For instance, they are not closed-symmetric monoidal, so one could spend ages wondering if a certain colimit commutes with a tensor product, et cetera. To an algebraist, these pathologies are an enormous headache, and they are fatal for the theory of locally-convex vector spaces.

Rather, the work of Ben-Bassat–Kelly–Kremnizer tells us which $\mathscr{V}$ to choose. Let us take the following as an axiom: there is nothing wrong with the category of Banach spaces, *besides* the fact that it only has finite limits and colimits. Indeed, Banach spaces have a closed monoidal structure, they are quasi-abelian (I will say more about this in a bit), and they have enough projectives, so they are a pretty good setting to do algebra.

To add these missing (co)limits in, we can take some kind of completion of the category. The question is which one to take. The insight of the above-mentioned authors, is that one should take an Ind-completion, whence it is totally obvious that one gets a closed symmetric monoidal category, by Ind-extending the tensor product.

If you think about it, the category of (complete) locally-convex vector spaces is like a Pro-completion of Banach spaces. In the same way that locally-convex vector spaces can be described as a a vector space equipped with a notion of "convex open subsets", complete bornological spaces (which are equivalent to monomorphic Ind-systems of Banach spaces), can be described as vector spaces with a notion of bounded subsets, satisfying a number of obvious properties. The theory of bornological spaces has a long history, going back to Mackey [Mac45], Waelbroeck [Wae67], Hogbe-Nlend [HN70, HN71], Houzel [Hou72], and more. In fact, the definition appears in Bourbaki [Bou81, Chapter III, §1].

Now let me explain why it is so critically important to "go derived" if we truly want analytic geometry to be on the same footing as algebraic geometry. From a module theorist's perspective, the most important notion in algebraic geometry is that of a quasi-coherent sheaf. In some sense, they are the dictionary between algebra and geometry. Their utility lies in the fact that they are local on the scheme X, which one can reinterpet as saying that QCoh is a sheaf of categories.

It is well-known due to the example of Gabber [Con06, Example 2.1.6] that the naïve definition of a quasi-coherent sheaf cannot satisfy descent in the analytic topology. The inherent pathology is the failure of flatness of localizations, with respect to the completed tensor product. To fix this, for our theory of quasicoherent sheaves we will need to glue categories of complexes of modules along derived localizations. However, mapping cones in triangulated categories fail to be functorial in general and so one cannot, for instance, define a "glued sheaf" along two open subsets to be the "fiber of the two restriction morphisms". This necessitates the introduction of higher algebra; instead of triangulated categories, we use stable ∞-categories. Now we can state "ansatz" which informs this work in a slightly more precise way:

Derived analytic geometry is algebraic geometry relative to $\mathscr{V} = D(\mathsf{CBorn}_R)$.

Here the $D(\mathsf{CBorn}_R)$ stands for the derived category of complete bornological R-modules, for some Banach ring[1] R, or rather its obvious ∞-categorical enhancement.

One might object and point out that CBorn_R is not an abelian category. The observation of Schneiders and Prosmans [Sch99, PS00a, Pro95] was that this is still a very special kind of exact category known as a quasi-abelian category, amenable to homological algebra. Further, in [Kel24], Kelly produced a projective model structure on unbounded chain complexes valued in certain exact categories. The combined work of Schneiders, Prosmans and Kelly solves the problem of how to do homotopical algebra in quasi-abelian categories[2].

Working with ∞-categories turns out to be an enormous advantage for a number of other reasons. For instance, we gain access to the powerful machinery of stable and presentable ∞-categories, the Barr–Beck–Lurie monadicity theorem, the theory of descendable algebras, Fourier–Mukai transforms (the list goes on). Because the language of ∞-categories is so uniform, there is a remarkable conservation of difficulty. Thanks to the breakthrough works of Liu–Zheng [LZ17], Gaitsgory–Rozenblyum [GR17], and Mann [Man22], a full six-functor formalism can be implemented with ease. In keeping with the above philosophy, most of these constructions work in the generality of *relative algebraic geometry*, and we get the corresponding constructions in analytic geometry just by specialising to the case when $\mathscr{V} = D(\mathsf{CBorn}_R)$.

Relation to condensed mathematics. As mentioned above, our functional analysis is based on bornological spaces. An alternative approach would have us use the condensed

[1] We remark that the initial object in the category of Banach rings is $(\mathbf{Z}, |\cdot|_\infty)$, or $(\mathbf{Z}, |\cdot|_{\text{triv}})$ in the category of non-Archimedean Banach rings. This raises the rather exciting possibility of a theory of "global analytic geometry", which *includes* the Archimedean place.

[2] A common misconception about this theory, is that one has to work with the abelian envelope when doing homological algebra, in effect losing some of the concreteness of the theory. Thanks to the existence of the model structures, this is not true.

mathematics of Clausen and Scholze [CS19a]. My understanding is that the thesis of Stempfhuber [Ste25] establishes a precise relation between the two theories.

1.2 What is done in Chapter 2.

Here is the organization of my thesis, and my results.

§2.1: Homotopical algebra in quasi-abelian categories. In this section we review the methods established in [Sch99] and [Kel24].

In §2.1.1 we establish our conventions on bornological spaces and recall various categorical and functional-analytic facts.

In §2.1.2 we collect various properties of the derived category $D(\mathscr{A})$ of a quasi-abelian category $\mathscr{A}$, generalizing the corresponding results in the case when $\mathscr{A}$ is abelian. These results may be well-known, but we find it convenient to have them written in one place, and using arguments which work in the appropriate generality, i.e., do not delve into the details of chain-complexes. Under the appropriate hypotheses, c.f. §2.1.2, we prove the following:

- ⋆ We characterise $D^{\leqslant 0}(\mathscr{A})$ as the *free sifted cocompletion* on a generating family of compact projectives (Corollary 2.1.39(i)).
- ⋆ We collect various *universal properties* of $D^{\leqslant 0}(\mathscr{A})$ and $D(\mathscr{A})$ (Corollary 2.1.39(ii) and Proposition 2.1.50). These essentially follow from the above and the formula $D^{\leqslant 0}(\mathscr{A}) \otimes \mathsf{Sp} \simeq D(\mathscr{A})$ (Lemma 2.1.48).
- ⋆ We prove that $D^{\leqslant 0}(\mathscr{A})$ and $D(\mathscr{A})$ are *compactly generated* and we give a description of the compact objects as *"$\mathscr{P}$-perfect complexes"* (Proposition 2.1.44 and Proposition 2.1.49)[3].

In §2.1.3 we investigate the monoidal structure on $D^{\leqslant 0}(\mathscr{A})$, in the case when $\mathscr{A}$ is closed symmetric monoidal. Under appropriate hypotheses, c.f. §2.1.3, we prove that:

- ⋆ The monoidal structure on $D^{\leqslant 0}(\mathscr{A})$ agrees with the Day convolution monoidal structure on the sifted cocompletion (Corollary 2.1.56).
- ⋆ Using this we formulate universal properties for $D^{\leqslant 0}(\mathscr{A})$ and $D(\mathscr{A})$ as *monoidal ∞-categories* (Corollary 2.1.56 and Proposition 2.1.57).

§2.2: Monads and descent. In order to formulate and prove the results in the appropriate generality, we decided to include this in a separate section.

In §2.2.1 we recall the definition of a homotopy-coherent monad acting on an ∞-category, together with a module over it, and the statement of the Barr–Beck–Lurie monadicity theorem.

In §2.2.2 we bootstrap the theorem of Barr–Beck–Lurie, to obtain a version which works in families. That is, we prove a version of Barr–Beck–Lurie which works for relative adjunctions between coCartesian fibrations (Proposition 2.2.5). This generalizes the result of [GHK22, Proposition 4.4.5]; it is possible that the hypotheses in Proposition 2.2.5 are optimal. By straightening, we obtain a "parametrized monadicity theorem" (Corollary 2.2.7).

[3] It is quite curious that, when we take $\mathscr{A} = \mathsf{CBorn}_K$, then certain computations in functional analysis boil down to this *categorical property* of compact generation: see for instance the proof of Lemma 3.2.38.

In §2.2.3 we formulate and prove a mild generalization of Mathew's theorem on descendable algebras [Mat16, §3.3] to the noncommutative setting.

In §2.2.4 we investigate the relation between *monoids* and *monads*. One might think that this question answered by the theory of Fourier–Mukai transforms (see §2.3.3), however, in the course of this thesis we will encounter functors which are not linear but only *lax linear*, and we still want to relate them to functors given by "tensoring with a bimodule". Here is the main result of §2.2.4:

Theorem 1.2.1 (= Corollary 2.2.22). *Let $\mathscr{V}$ be a monoidal ∞-category and let $A \in \mathsf{Alg}(\mathscr{V})$ be an algebra object in $\mathscr{V}$. There is an adjunction*

$$\iota : {}_A\mathrm{BMod}_A\,\mathscr{V} \leftrightarrows \mathrm{Fun}^{\mathrm{lax}}_{\mathscr{V}}(\mathrm{RMod}_A\,\mathscr{V}, \mathrm{RMod}_A\,\mathscr{V}) : \kappa \tag{1.1}$$

*in which the left adjoint ι is strongly monoidal (for convolution on the left, and composition on the right). Here the superscript "*lax*" denotes the* lax $\mathscr{V}$-linear *functors.*

In particular the right adjoint κ is lax-monoidal. The intended application is the following. Suppose we are given a monad T belonging to the right-hand side of (1.1), meaning that its underlying endofunctor is lax $\mathscr{V}$-linear (such arises quite naturally from adjunctions in which the left adjoint is $\mathscr{V}$-linear). Then, according to the Theorem, $T(A)$ becomes an algebra object under convolution, and the counit $\iota\kappa \to \mathrm{id}$ furnishes a morphism of monads

$$T(A) \otimes_A (-) \to T(-), \tag{1.2}$$

which one might call the "best linear approximation".[4] Thus, the problem of relating modules over the monad T to modules over the algebra $T(A)$ reduces to understanding "fixed points": that is, those objects where the natural transformation (1.2) resticts to an an equivalence. We will apply this method in the context of analytic $\mathcal{D}$-modules in §3.2.6.

§2.3: Theory of abstract six-functor formalisms. It is very convenient for a number of reasons to construct a six-functor formalism. The idea of a *six-functor formalism* is that we start with some ∞-category $\mathscr{C}$ of geometric objects X, admitting all fiber products, and associate to each $X \in \mathscr{C}$ a closed symmetric monoidal ∞-category $(Q(X), \otimes)$ in a manner which satisfies an enormous number of functorial properties. That is, to each morphism f of $\mathscr{C}$ we associate a *pullback* and a *pushforward* functor, and for morphisms in a certain special class E we associate a *compactly supported pushforward* and an *exceptional pullback*. These assignments should all be compatible with composition, and satisfy the *base-change* and *projection* formulas[5]. In recent years, due to the breakthrough works of Liu–Zheng [LZ17], Gaitsgory–Rozenblyum [GR17], and Mann [Man22], our collective understanding of the six operations has developed from a *yoga* to a rigorously defined notion. Roughly speaking, on associates to the pair $(\mathscr{C}, E)$ the *category of correspondences* $\mathrm{Corr}(\mathscr{C}, E)$, whose objects are the same as $\mathscr{C}$ and whose morphisms are given by spans. A six-functor formalism is then a lax-symmetric monoidal functor $Q : \mathrm{Corr}(\mathscr{C}, E) \to \mathsf{Cat}_\infty$[6]. This amazingly succinct definition provides a viable way to manipulate six-functor formalisms and produce new ones out of old ones.

In §2.3 we recall some fundamental notions in the theory of six-functor formalisms, following the conventions established by Mann [Man22, §A.5].

[4] This is not related to the notion of "best linear approximation" in the Goodwillie calculus.

[5] For a more detailed exposition we direct the reader to §2.3 and [Man22, Sch22].

[6] Once again, for a more detailed exposition we direct the reader to §2.3 and [Man22, Sch22].

In §2.3.1 we develop an *extension formalism* for abstract six-functor formalisms. The content of §2.3.1 is mostly a re-hashing of [Sch22, Theorem 4.20] and we include it to convince the reader that the result of *loc. cit.* is true in greater generality (please see also Remark 2.3.7). Although it is rather technical, the basic idea is the following. In many situations, one can often construct a basic six-functor formalism; for instance, in analytic geometry, it is not so hard to construct a six-functor formalism for quasi-coherent sheaves in which the !-able morphisms are the qcqs ones, and they all satisfy $f_! = f_*$. However, this often doesn't include most of the interesting morphisms for which we want to define !-functors: for instance, in analytic geometry, many morphisms are not quasi-compact. Therefore, we should enlarge the class of !-able morphisms. This is the purpose of Theorem 2.3.10, which allows us to enlarge the class of !-able morphisms to a class E with good stability properties: namely E is !-local on the source, $*$-local on the target, stable under disjoint unions and tame (Definition 2.3.9). A heuristic is that this extension formalism is giving us a purely categorical (as opposed to geometric) way to arrive at a notion of "relative compact supports".

In §2.3.2 we apply the extension formalism of §2.3.1 to obtain a six-functor formalism for quasi-coherent sheaves on (derived) stacks in *relative algebraic geometry* (Theorem 2.3.17). It seems more appropriate to direct the reader to §2.3.2, but we remark that Theorem 2.3.17 is directly applied to construct the six-functor formalism for quasi-coherent sheaves on derived rigid spaces in §3.1.5.

In §2.3.3 we investigate the theory of Fourier–Mukai transforms in relative algebraic geometry. Our focus is not on defining Fourier–Mukai transforms, given that this problem was completely solved[7] in [HM24]. Rather, given morphisms of stacks $X \to Y \leftarrow Z$ in relative algebraic geometry, we investigate the following questions:

- $\star$ When does the tensor product formula
$$\mathrm{QCoh}(X) \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(Z) \simeq \mathrm{QCoh}(X \times_Y Z) \tag{1.3}$$
hold?
- $\star$ When is $\mathrm{QCoh}(X)$ dualizable and canonically self-dual as a $\mathrm{QCoh}(Y)$-module category?
- $\star$ When is the Fourier–Mukai transform
$$FM : \mathrm{QCoh}(X \times_Y Z) \to \mathrm{Fun}^L_{\mathrm{QCoh}(Y)}(\mathrm{QCoh}(X), \mathrm{QCoh}(Z)) \tag{1.4}$$
an equivalence of ∞-categories?

Our answers to these questions are contained in Theorem 2.3.22 and Lemma 2.3.23. I am grateful to Peter Scholze for explaining the main idea of the proof to me. As usual, by specializing to $\mathscr{V} = D(\mathsf{CBorn}_R)$ we obtain the corresponding statements in derived analytic geometry: see Corollary 3.1.47 for a sample application.

1.3 What is done in Chapter 3.

Let $K/\mathbf{Q}_p$ be a complete field extension.

[7] That is, Heyer–Mann defined the 2-functor from their 2-category of correspondences to Pr^L (in very great generality), which gives the Fourier–Mukai transform *together with* its compatibility with composition.

§3.1: Derived rigid geometry. As the reader has probably gathered by now, the idea of §3.1 is that by doing algebraic geometry relative to $D(\mathsf{CBorn}_K)$, we obtain a theory of derived analytic geometry, which contains a theory of derived rigid geometry. The material of §3.1 owes an overwhelming intellectual debt to [BBKK24] and also takes much inspiration from [Man22, §2].

In §3.1.1 we define the category $\mathsf{dAfndAlg}$ as a certain full subcategory of the monoids in $D_{\geqslant 0}(\mathsf{CBorn}_K)$. For any $A \in \mathsf{dAfndAlg}$, its truncation $\pi_0 A$ is an affinoid algebra in the classical sense. We define dAfnd to be the opposite ∞-category to $\mathsf{dAfndAlg}$. We denote the object of dAfnd corresponding to $A \in \mathsf{dAfndAlg}$ by the formal expression $\mathrm{dSp}(A)$. We define the *weak Grothendieck topology* on dAfnd whose covers are essentially given by finite jointly-surjective collections of *derived rational subspaces*. We prove that this topology is subcanonical and that the prestack sending

$$\mathrm{dSp}(A) \mapsto \mathrm{QCoh}(\mathrm{dSp}(A)) := \mathrm{Mod}_A(D(\mathsf{CBorn}_K)) \tag{1.5}$$

is a sheaf in the weak topology.

In §3.1.2 we define the category dRig of derived rigid spaces as a certain full subcategory of $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd}, \infty\mathsf{Grpd})$. The full subcategory dRig is closed under all coproducts and fiber products, and dRig is equipped with the *strong Grothendieck topology* whose covers are given by jointly-surjective families of *analytic subspaces*. By right Kan extension along $\mathsf{dAfnd}^{\mathsf{op}} \to \mathsf{dRig}^{\mathsf{op}}$, the functor QCoh becomes a sheaf in the analytic topology.

In §3.1.3 we explain how Hoffman-Lawson duality [Joh86, VII §4] can be used to associate a locally spectral topological space $|X|$ to any $X \in \mathsf{dRig}$, whose locale of open subsets is canonically isomorphic to the locale of analytic subspaces of X. We also define a functor $X \mapsto X_0$ sending X to its *classical truncation*, which extends $\mathrm{dSp}(A) \mapsto \mathrm{dSp}(\pi_0 A)$, and prove the topological invariance property $|X| \cong |X_0|$. Thus, derived rigid geometry obeys one of the principles of derived geometry, that "all the geometry happens on X_0".

In §3.1.4 we define qcqs morphisms of derived rigid spaces and prove some of their properties. For any morphism $f : X \to Y$ in dRig, we write f^* for the symmetric-monoidal pullback functor from $\mathrm{QCoh}(Y)$ to $\mathrm{QCoh}(X)$ and f_* for its right adjoint. In Lemma 3.1.37, we show that for any qcqs morphism $f : X \to Y$ in dRig, the functor f_* satisfies base-change and the projection formula. This allows for the construction of a basic six-functor formalism in which the !-able morphisms are the qcqs ones (Corollary 3.1.39).

In §3.1.5 we investigate further the six-functor formalism for quasi-coherent sheaves on derived rigid spaces. We prove in Proposition 3.1.44 that the six-functor formalism constructed in §3.1.4, is the unique extension from $(\mathsf{dAfnd}, \mathrm{all})$ to $(\mathsf{dRig}, \mathrm{qcqs})$. On the other hand, by the formalism of §2.3.2 we obtain a six-functor formalism on $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd}, \infty\mathsf{Grpd})$, extending the one on $(\mathsf{dAfnd}, \mathrm{all})$, with a class of !-able edges E which has all the good stability properties. By combining this with the unicity, we obtain the desired six-functor formalism on dRig with a much larger class of !-able edges than just the qcqs ones.

In §3.1.6 we establish that various interesting (non quasi-compact) morphisms in rigid geometry, really do belong to the class E of !-able morphisms. Using that the class E is !-local on the source, this boils down to showing that certain infinite covers are of universal !-descent (Corollary 3.1.51), so that for instance the morphism $\mathbf{A}^1_K \to *$ is !-able (Example 3.1.52). A notable feature of our approach is that we do not use compactifications, only the notion of "compact supports" provided to us by §2.3.1.

In §3.1.7 we develop a theory of local (co)homology in derived rigid geometry. Let $X \in \mathsf{dRig}$ and $S \subseteq |X|$ be a closed subset. Under the hypothesis that the complementary open $j : U \hookrightarrow X$ satisfies $j^! \simeq j^*$, we obtain various recollement sequences (Proposition 3.1.53). In Proposition 3.1.54 we use the results of §3.1.6 to provide a criterion for when

$j^! \simeq j^*$. As a by-product we also obtain formulas for the local (co)homology functors in terms of sequential limits or colimits.

In §3.1.8 and §3.1.9 we define Zariski-closed and Zariski-open immersions as the complements of Zariski-closed immersions. We show in Proposition 3.1.60 that these fit in to the formalism of §3.1.7: in particular, if $j : U \to X$ is a Zariski-open immersion, there is an equivalence $j^! \simeq j^*$.

In §3.1.10 we introduce germs along Zariski-closed immersions. The definition is quite simple. Let $i : Z = \mathrm{dSp}(B) \to \mathrm{dSp}(A) = X$ be a Zariski-closed immersion of derived affinoids, induced by a morphism $A \to B$ surjective on π_0. Then the *germ along* Z is

$$A_Z^\dagger := \operatorname*{colim}_{U \supseteq |Z|} A_U \tag{1.6}$$

where the colimit is taken in $\mathsf{dAlg} := \mathsf{CAlg}(D_{\geqslant 0}(\mathsf{CBorn}_K))^{\mathsf{op}}$ and runs over all affinoid subdomains $U \supseteq |Z|$. We denote the object of the opposite category $\mathsf{dAff} := \mathsf{dAlg}^{\mathsf{op}}$ corresponding to $A_Z^\dagger$ by the formal expression $(Z \subseteq X)^\dagger$. In Proposition 3.1.64 we show that there is a natural equivalence of ∞-categories

$$\Gamma_Z \operatorname{QCoh}(X) \simeq \operatorname{QCoh}((Z \subseteq X)^\dagger) = \operatorname{Mod}_{A_Z^\dagger} D(\mathsf{CBorn}_K), \tag{1.7}$$

in algebra, we are familiar with the identification between "sheaves with support" and sheaves on the formal completion, and this is nothing but an analytic counterpart to that. Indeed, a recurring theme of this thesis is the following: everywhere where one would see a "formal neighbourhood" in algebraic geometry, in our analytic geometry we instead replace it by an *analytic germ*, and see what we get.

In the brief §3.1.11 we investigate a six-functor formalism which incorporates the above "germs". In this we prefer a naïve approach. That is, we take the "trivial" six-functor formalism on $\mathsf{dAff} = \mathsf{CAlg}(D_{\geqslant 0}(\mathsf{CBorn}_K))$ which sends $X = \mathrm{dSp}(A)$ to $\operatorname{QCoh}(\mathrm{dSp}(A)) := \operatorname{Mod}_A D(\mathsf{CBorn}_K)$ and in which every morphism f satisfies $f_! = f_*$. The utility is that our "germs" naturally belong to dAff. Then we apply the formalism of 2.3.1 to obtain a six-functor formalism on $\mathsf{PStk} := \operatorname{Psh}(\mathsf{dAff}, \infty\mathsf{Grpd})$, with a class of !-able edges which has the good stability properties.

§3.2: Stratifications and analytic $\mathcal{D}$-modules. The theory developed in §3.1.10 and §3.1.11 allows us in §3.2.2 to contemplate the following. For any morphism $f : X \to Y$ in dAfnd and any $n \geqslant 0$ we can consider the germ $(X \subseteq X^{n+1/Y})^\dagger$ along the diagonal. Letting n vary, these can be arranged into a simplicial object in PStk, which is in fact an *internal groupoid object*, called the *infinitesimal groupoid* and denoted $\operatorname{Inf}(X/Y)$. We define the *stratifying stack* of f as the geometric realization

$$(X/Y)_{\mathrm{str}} := \operatorname*{colim}_{[n] \in \Delta^{\mathsf{op}}} \operatorname{Inf}(X/Y)_n, \tag{1.8}$$

where the colimit is taken in PStk. When $Y = \mathrm{dSp}(K) = *$ is the terminal object, we just write $\operatorname{Inf}(X)$ and X_{str}. As in [Sch22, Lecture VIII], the idea is that the six-functor formalism for stratifications already lives in the six-functor formalism on PStk. More precisely, we identify a class of *good* morphisms, c.f. Definition 3.2.13, which is stable under base-change and composition, and is such that $(-)_{\mathrm{str}}$ induces a functor

$$(-)_{\mathrm{str}} : \operatorname{Corr}(\mathsf{dAfnd}, \mathrm{good}) \to \operatorname{Corr}(\mathsf{PStk}, \widetilde{E}), \tag{1.9}$$

where we have denoted the class of !-able edges in the six-functor formalism on PStk by $\widetilde{E}$. By post-composition with the six-functor formalism QCoh on $(\mathsf{PStk}, \widetilde{E})$ gives a basic six-functor formalism

$$\mathrm{Strat} := \mathrm{QCoh} \circ (-)_{\mathrm{str}} \tag{1.10}$$

for *analytic crystals*. By Kan extension, we can lift Strat to a six-functor formalism on all of dRig, in which the class E_{str} of !-able morphisms contains all those which are representable in the class *good*. Unfortunately, it does not seem that the extension formalism of §2.3.1 applies here because the class of *good* morphisms does not have the right-cancellative property, so that the extension principles of [Man22] have to be applied in a more ad-hoc way.

In §3.2.3 we prove that Strat, when viewed as a prestack via the upper-star functors, is a sheaf in the analytic topology (Lemma 3.2.17). We prove a version of Kashiwara's equivalence for a class of Zariski-closed immersions $i : Z \to X$ in dRig which we call *stratifying*: this means that i locally admits a retraction.

In §3.2.4, we investigate the relation between $\mathrm{Strat}(X)$ and "$\mathcal{D}$-modules". Let $X \in \mathsf{dAfnd}$. The key to understanding the relation is looking at the canonical morphism $p : X \to X_{\mathrm{str}}$ and the induced adjunctions $p^* \dashv p_*$ and $p_! \dashv p^!$ on quasi-coherent sheaves. We define the *comonad of (analytic) jets* to be $\mathcal{J}_X^\infty := p^*p_*$ and the *monad of (infinite-order) differential operators* to be $\mathcal{D}_X^\infty := p^!p_!$. It is always the case that $p^* \dashv p_*$ is comonadic: that is, there is an equivalence between $\mathrm{Strat}(X)$ and comodules in $\mathrm{QCoh}(X)$ over the comonad $\mathcal{J}_X^\infty$. If $X \to X_{\mathrm{str}}$ is *of !-descent* then the adjunction $p_! \dashv p^!$ is monadic so that there is an equivalence between $\mathrm{Strat}(X)$ and the category of modules over the monad $\mathcal{D}_X^\infty$. It turns out that there is a canonical equivalence $p_! \simeq p_*$ and that this can be used to pass between these descriptions in terms of $\mathcal{J}_X^\infty$-comodules and $\mathcal{D}_X^\infty$-modules. By base-change, the underlying endofunctors of $\mathcal{J}_X^\infty$ and $\mathcal{D}_X^\infty$ are given by the simple formulas

$$\mathcal{J}_X^\infty \simeq \tilde{\pi}_{1,*}\tilde{\pi}_2^* \simeq (A\widehat{\otimes}_K A)^\dagger_\Delta \widehat{\otimes}_A (-) \quad \text{and} \quad \mathcal{D}_X^\infty \simeq \tilde{\pi}_{2,*}\tilde{\pi}_1^! \simeq R\underline{\mathrm{Hom}}_A((A\widehat{\otimes}_K A)^\dagger_\Delta, -).$$

Here $\tilde{\pi}_1, \tilde{\pi}_2 : (X \subseteq X \times X)^\dagger \to X$ are the two projections. Moreover, under suitable hypotheses, we can chase the explicit equivalence of categories implicit in the Barr–Beck–Lurie theorem to give formulas for the six operations in Theorem 3.2.28. For some reason, it turns out to be convenient to use *both* the descriptions to give these formulas, in terms of jets *and* differential operators.

In Theorem 3.2.34, we give a partial answer to the question of when the morphism $p : X \to X_{\mathrm{str}}$ is of (universal) !-descent. From the previous discussion, this is clearly important for knowing when $\mathrm{Strat}(X)$ is related to $\mathcal{D}_X^\infty$-modules. We prove that p is of universal !-descent whenever X is a classical affinoid equipped with an étale morphism $X \to \mathbf{D}_K^r$.

In §3.2.6 we use the method of §2.2.4 to relate modules over the monad $\mathcal{D}_X^\infty$ to modules over Ardakov–Wadsley's ring $\widehat{\mathcal{D}}_X(X)$. Here are our main results: as it turns out, the condition that the natural transformation (1.2) is an equivalence, is related to the functional-analytic property of being Fréchet.

Theorem 1.3.1. *Let $X = \mathrm{Sp}(A)$ be a classical affinoid equipped with an étale morphism $X \to \mathbf{D}_K^r$. Let $\mathrm{Fr}(X) \subseteq \mathrm{QCoh}(X)$ be the full subcategory spanned by complexes whose cohomology groups are Fréchet spaces. Then:*

(i) The full subcategory $\mathrm{Fr}(X)$ is preserved by the monad $\mathcal{D}_X^\infty$ and there is an equivalence of ∞-categories

$$\mathrm{RMod}_{\widehat{\mathcal{D}}_X(X)} \mathrm{Fr}(X) \simeq \mathrm{Mod}_{\mathcal{D}_X^\infty} \mathrm{Fr}(X). \tag{1.11}$$

(ii) The category $D_{\mathcal{C}}(X)$ of Bode's $\mathcal{C}$-complexes[8] *[Bod21, §8] is naturally a full subcategory of* $\mathrm{RMod}_{\widehat{\mathcal{D}}_X(X)} \mathrm{Fr}(X)$*. In particular, by the result of Theorem 3.2.34, we obtain a fully-faithful functor*

$$D_{\mathcal{C}}(X) \hookrightarrow \mathrm{Strat}(X) = \mathrm{QCoh}(X_{\mathrm{str}}). \tag{1.12}$$

Since the heart of $D_{\mathcal{C}}(X)$ is the category of coadmissible $\widehat{\mathcal{D}}_X(X)$-modules, this gets us a fully-faithful functor from coadmissible $\widehat{\mathcal{D}}_X(X)$-modules to $\mathrm{Strat}(X)$. So they really do have an interpretation in terms of "†-infinitesimal parallel transport".

In §3.2.7 we prove various descent results for $\widehat{\mathcal{D}}$-modules. One important input is the noncommutative notion of descendability from §2.2.3. We recall the definition of the Banach-completed differential operators $\mathcal{D}_X^n(X)$ from, for instance [Bod21, §2]; these are Noetherian Banach algebras and one has $\widehat{\mathcal{D}}_X(X) = \lim_n \mathcal{D}_X^n(X)$.

Theorem 1.3.2. *Let $X = \mathrm{Sp}(A)$ be a smooth classical affinoid equipped with an étale morphism $X \to \mathbf{D}_K^r$.*

(i) The prestacks

$$\mathrm{RMod}_{\widehat{\mathcal{D}}_X(-)} D(\mathsf{CBorn}_K) \quad \textit{and} \quad \mathrm{RMod}_{\mathcal{D}_X^n(-)} D(\mathsf{CBorn}_K) \tag{1.13}$$

are sheaves in the weak topology on X.

(ii) The prestack $\mathrm{RMod}^{\mathrm{b,fg}}_{\mathcal{D}_X^n(-)} D(\mathsf{CBorn}_K)$ is a sheaf on the site of p^n-accessible[9] *affinoid subdomains of X. Here the superscript* b, fg *denotes cohomologically bounded complexes with finitely-generated cohomology.*

(iii) There is an equivalence of ∞-categories

$$D_{\mathcal{C}}(X) \simeq \lim_n \mathrm{RMod}^{\mathrm{b,fg}}_{\mathcal{D}_X^n(-)} D(\mathsf{CBorn}_K), \tag{1.14}$$

where the left-hand-side again denotes Bode's $\mathcal{C}$-complexes [Bod21, §8].

(iv) The prestack $D_{\mathcal{C}}(-)$ is a sheaf in the weak topology on X.

I am grateful to Andreas Bode for helpful discussions about the proof of Theorem 1.3.2(iii).

It is natural to ask whether the embedding of $\mathcal{C}$-complexes in (1.12) is compatible with restrictions, so that it can be globalised. This turns out to be true (that is Theorem 3.2.70) but the proof is non-trivial for the following reason: the comparison between $\mathrm{Strat}(X)$ and $\mathcal{D}^\infty$-modules comes from forgetting via an *upper-shriek* functor, but we want to use the *upper-star* restriction functors for $\widehat{\mathcal{D}}$-modules. In fact, we have to use the "parametrized monadicity theorem" of §2.2.2 to manage some of the coherences. Using this compatibility with restrictions we obtain the following.

Theorem 1.3.3. *Let X be a smooth classical rigid space. For such X we define the value of $D_{\mathcal{C}}(-)$ by Kan extension*[10]*, see Definition 3.2.71. Then there is a fully-faithful functor*

$$D_{\mathcal{C}}(X) \hookrightarrow \mathrm{Strat}(X) = \mathrm{QCoh}(X_{\mathrm{str}}). \tag{1.15}$$

This induces a fully-faithful functor

$$\{\textit{coadmissible } \mathcal{D}_X\textit{-modules}\} \hookrightarrow \mathrm{Strat}(X). \tag{1.16}$$

[8] To be precise, I mean its natural ∞-categorical enhancement.

[9] By this we mean $p^n\mathcal{T}_X$-accessible in the sense of [AW19, §4.5].

[10] If you like, you could call this the "∞-stackification" of Bode's $\mathcal{C}$-complexes.

In the future we may try to give a more intrinsic characterisation of the essential image of these functors. This is not completely straightforward, as Proposition 3.2.75 shows that the essential image of (1.15) is not contained in the dualizable objects.

Relation to existing theories. The content of Chapter 3 is similar in spirit to that of Rodríguez Camargo [Cam24]. We believe that there should be a precise relation between our theory and his. This is of course related to the problem of compatibility of the condensed and bornological formalisms of analytic stacks.

On the potential applications. We were originally motivated by applications to the study of locally-analytic representations of p-adic Lie groups. Let G be a rigid-analytic group. Then the germ $(1 \subseteq G)^\dagger$ along the unit of G is a group object in PStk. Hence, $(1 \subseteq G)^\dagger$ may act on objects of PStk, in particular, suppose that $(1 \subseteq G)^\dagger$ acts on a rigid variety X. In this situation we may form the quotient

$$X/(1 \subseteq G)^\dagger := \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} X \times (1 \subseteq G^n)^\dagger. \tag{1.17}$$

the colimit being taken in PStk. The action morphism determines a canonical morphism of groupoid objects:

$$X \times (1 \subseteq G^\bullet)^\dagger \to (X \subseteq X^{\bullet+1})^\dagger, \tag{1.18}$$

and taking colimits we obtain a correspondence in PStk:

$$X_{\mathrm{str}} \xleftarrow{\beta} X/(1 \subseteq G)^\dagger \xrightarrow{\alpha} */(1 \subseteq G)^\dagger. \tag{1.19}$$

A slogan for the morphism β is that $(1 \subseteq G)^\dagger$ acts on X by "†-infinitesimal symmetries". Similarly to the situation for X_{str}, one has that $\mathrm{QCoh}(*/(1 \subseteq G)^\dagger)$ can be described as comodule objects in $\mathrm{QCoh}(*)$ over the Hopf algebra object $C_1(G, K)$ of germs of functions at the identity[11].

Such objects arise naturally in the following context. Assume that $\mathbf{Q}_p \subseteq L \subseteq K$ is an intermediate field extension with $L/\mathbf{Q}_p$ finite. Let $\mathbf{G}$ be a reductive algebraic group defined over L. Then $\mathbf{G}(L)$ can be regarded as a locally L-analytic group which we denote by $\mathbf{G}(L)^{\mathrm{la}}$. If we set $G = \mathbf{G}_K^{\mathrm{rig}}$, that is, the rigid-analytification, then one can identify the germs of functions at the identity:

$$C_1(G, K) = C_1(\mathbf{G}(L)^{la}, K). \tag{1.20}$$

Therefore, the restriction of locally-analytic representations to the germ at 1, is a source of objects in $\mathrm{QCoh}(*/(1 \subseteq G)^\dagger)$. For more about locally-analytic representations we refer the reader to the works of Schneider–Teitelbaum [ST02, ST03]. In the condensed setting, Rodríguez Camargo–Rodrigues Jacinto have also given a stacky interpretation of locally-analytic representations [RJRC22, RJRC25].

Returning to the correspondence (1.19), the six-functor formalism on PStk furnishes an adjunction

$$\beta_!\alpha^* : \mathrm{QCoh}(*/(1 \subseteq G)^\dagger) \leftrightarrows \mathrm{QCoh}(X_{\mathrm{str}}) : \alpha_*\beta^!. \tag{1.21}$$

The left adjoint $\beta_!\alpha^*$ here is the analytic Beilinson–Bernstein localization functor. In the case when $G = \mathbf{G}_K^{\mathrm{rig}}$ as above, and $X = (\mathbf{G}_K/\mathbf{B}_K)^{\mathrm{rig}}$ is the flag variety, with its

[11] When the underlying object of $\mathrm{QCoh}(*)$ belongs to $\mathrm{Fr}(*)$, one also has a relation to module objects over the Arens–Michael envelope $\widehat{U(\mathfrak{g})}$, in a similar manner to Theorem 1.3.1.

natural $(1 \subseteq G)^\dagger$-action obtained by restricting the natural G-action, one expects the adjunction (1.21) to be close to an equivalence. That is, if one restricts to objects of $\mathrm{QCoh}(*/(1 \subseteq G)^\dagger)$ with trivial infinitesimal character, then (1.21) should be an equivalence of ∞-categories.

In the algebraic setting, this stacky approach to Beilinson–Bernstein localization has been carried out by Ben-Zvi–Nadler [BZN19], and in a real-analytic setting (using condensed mathematics) by Scholze [Sch24]. In our setting, it is possible that this analytic Beilinson–Bernstein localization recovers and extends the result of Ardakov [Ard21, Theorem 6.4.8], removing the finiteness condition of (co)admissibility.

1.4 What is done in Chapter 4

In Chapter 4 we investigate the possibility that the material of Chapter 3 could be simplified and improved via the use of *algebraic theories*. Chapter 4 was written *after* the results of the rest of this thesis were obtained, and consequently it is independent of the results of Chapter 3. We include it because it seems interesting.

Many kinds of "geometry" are constructed in the following way: one starts with some class of "free" algebras (usually corresponding to some kinds of functions on "Cartesian spaces" like $\mathbf{R}^n$) and enlarges it by sifted colimits to obtain the relevant category of algebras. The category of affine spaces is opposite to this category of algebras, and more general spaces can be interpreted as sheaves on this class of affines (which is like a kind of cocompletion).

This perspective is made rigorous via the following definition: A *Lawvere theory* is a (small) category $\mathscr{T}$ equipped with finite products, with a finite-product preserving, identity-on-objects[12] functor $\mathscr{T} \to \mathsf{Fin}$ to the category of finite sets. The category of algebras for $\mathscr{T}$ is[13] $\mathsf{Alg}_{\mathscr{T}} := \mathrm{Fun}^{\Pi}(\mathscr{T}, \mathsf{Set})$ and the obvious homotopical version of this notion is $\mathsf{dAlg}_{\mathscr{T}} := \mathrm{Fun}^{\Pi}(\mathscr{T}, \infty\mathsf{Grpd})$. It is tautological, and yet striking, to observe that

$$\mathsf{dAlg}_{\mathscr{T}} = \mathrm{sInd}(\mathscr{T}^{\mathsf{op}}), \tag{1.22}$$

agrees with the ∞-categorical sifted cocompletion, which is also Quillen's *nonabelian derived category* or the *animation* (in more modern nomenclature) of $\mathsf{Alg}_{\mathscr{T}}$. The category prestacks for $\mathscr{T}$ is $\mathrm{Psh}(\mathsf{dAlg}_{\mathscr{T}}, \infty\mathsf{Grpd})$.

By incorporating a set of *sorts* one arrives at the notion of a *multisorted Lawvere theory* which, for our purposes, captures the notion that our building-blocks are disks of *varying radius*: this appears to be the correct way to formulate rigid- and dagger-geometry via algebraic theories.

There is an extensive literature on this approach to (derived) geometry, which I have to admit that I am not fully up to speed with. Some sources with plenty of examples are the awesome paper of Carchedi and Roytenberg [CR13] and the recent work of Ben-Bassat–Kelly–Kremnizer [BBKK24].

In §4.1 we give a construction of a six-functor formalism, for quasi-coherent sheaves on prestacks relative to a Lawvere theory. We take a (presentably symmetric monoidal, stable) ∞-category $\mathscr{V}$ and we suppose that we are given a finite-coproduct preserving functor

$$\mathscr{T}^{\mathsf{op}} \to \mathsf{CAlg}(\mathscr{V}) \tag{1.23}$$

[12] Here we really mean the identity.

[13] Here and elsewhere, Fun^{Π} denotes the *finite* product-preserving functors.

we call this a *$\mathscr{V}$-realization of $\mathscr{T}$* (Definition 4.1.5). There is something of an art in finding a $\mathscr{V}$-realization: in many situations, it appears that taking $\mathscr{V} = D(\mathsf{CBorn}_R)$ for an appropriate Banach ring R works, and the functor (1.23) expresses that the free algebras are endowed with a canonical bornology. That (1.23) is finite-coproduct preserving often follows from flatness of the free algebras, with respect to the completed tensor product. In the presence of a $\mathscr{V}$-realization, we obtain a six-functor formalism for quasi-coherent sheaves on the category of prestacks for $\mathscr{T}$ (Theorem 4.1.6).

In §4.2 we obtain a six-functor formalism for "$\mathcal{D}^\infty$-modules" on prestacks relative to a *Fermat theory*. While we do not define it in this introduction (it is Definition 4.2.1), the reader should think of a Fermat theory as the minimal enhancement of a Lawvere theory needed to define certain structures of "differentiable nature", such as 1-forms, Taylor series, or cotangent complexes. The insight of [Tar25] is that we can perhaps add another item to this list: that is, a Fermat theory is the essentially the minimal structure needed to define an *analytic de Rham space*.

Let me elaborate. Fermat theories were originally introduced in [DK84] where many of their fundamental properties were proved. Possibly the most important is the following [DK84, Proposition 1.2]; any algebra A for a Fermat theory $\mathscr{T}$ has an underlying ring, and for any ideal I in the underlying ring, then A/I acquires the canonical structure of a $\mathscr{T}$-algebra. Using this one can write down various ideals, in particular one can essentially copy the definition of ∞-nilradical introduced in [BK18] to obtain an analytic notion[14] of reduction [Tar25, Construction A.33]. By precomposition with this analytic reduction functor, one obtains the analytic de Rham space $X_{\mathscr{T}\mathrm{dR}}$ of a prestack X (Definition 4.2.6) and a corresponding endofunctor $(-)_{\mathscr{T}\mathrm{dR}}$ of the category of prestacks. By pre-composing the six-functor formalism of §4.1 with the functor induced by $(-)_{\mathscr{T}\mathrm{dR}}$ we obtain a six-functor formalism for "$\mathcal{D}^\infty$-modules" for the Fermat theory $\mathscr{T}$.

This conjecturally gives a uniform construction of a six-functor formalism for "$\mathcal{D}^\infty$-modules" in (derived) smooth geometry, Stein geometry, rigid geometry, dagger geometry (over an Archimedean or non-Archimedean base): there are many possibilities. In complex geometry, a theory of $\mathcal{D}^\infty$-modules based on Ind-Banach spaces has been developed by Prosmans–Schneiders [PS00b]; it is possible that our construction yields a six-functor formalism in their setting.

The construction of Chapter 4 is sufficiently general that it can be used to obtain a six-functor formalism for "$\mathcal{D}^\infty$-modules" over the Banach ring $(\mathbf{Z}, |\cdot|_\infty)$. In future work we will explore these examples in more detail.

[14] In the case of a sorted Fermat theory, the definition of reduction has to be changed slightly: see Remark 4.2.7.

Chapter 2

Relative algebraic geometry

2.1 Homotopical algebra in quasi-abelian categories

We make extensive use of the theory of homotopical algebra in quasi-abelian categories as developed in [Kel24] and [Sch99]. We assume familiarity with the basics of higher algebra and model categories, as these topics are too vast to give a proper summary; we will give an indication of any non-standard or particular notions.

Let $\mathscr{A}$ be a quasi-abelian category. Recall [Sch99, §1] that this means that $\mathscr{A}$ is an additive category which has all kernels and cokernels, and strict[1] epimorphisms (resp. monomorphisms) are stable under pullbacks (resp. pushouts). We recall the following properties.

Definition 2.1.1. *(i) A functor $F : \mathscr{A} \to \mathscr{B}$ between quasi-abelian categories is called* left exact *(resp.* strongly left exact*) if it preserves the kernels of strict morphisms (resp. all morphisms).*

(ii) A functor $F : \mathscr{A} \to \mathscr{B}$ between quasi-abelian categories is called right exact *(resp.* strongly right exact*) if it preserves the cokernels of strict morphisms (resp. all morphisms).*

(iii) A functor $F : \mathscr{A} \to \mathscr{B}$ between quasi-abelian categories is called exact *(resp.* strongly exact*) if it is left and right exact (resp. strongly left and right exact).*

Definition 2.1.2. *[Kel24, Definition 2.47]*

(i) An object $P \in \mathscr{A}$ is called projective *if the functor* $\mathrm{Hom}(P, -) : \mathscr{A} \to \mathsf{Ab}$ *(valued in abelian groups), takes strict epimorphisms to surjections.*

(ii) We say that $\mathscr{A}$ has enough projectives *if for each object $M \in \mathscr{A}$ there exists a projective object P together with a strict epimorphism $P \twoheadrightarrow M$.*

Definition 2.1.3. *[Kel24, Definition 2.92] Assume that $\mathscr{A}$ admits small coproducts. A small subcategory $\mathscr{P}$ of objects in $\mathscr{A}$ is called* generating *if for each object $M \in \mathscr{A}$ there exists a small collection $\{P_i\}_{i\in\mathcal{I}}$ of objects of $\mathscr{P}$ together with a strict epimorphism $\bigoplus_{i\in\mathcal{I}} P_i \twoheadrightarrow M$.*

[1] Recall that a morphism f is called strict if the natural morphism $\operatorname{coker}\ker f \to \ker\operatorname{coker} f$ is an isomorphism.

Definition 2.1.4. *[Sch99, Definition 2.1.10]. Let $\mathscr{A}$ be a quasi-abelian category.*

(i) Assume that $\mathscr{A}$ admits (small) coproducts. An object $C \in \mathscr{A}$ is called small *if $\mathrm{Hom}_{\mathscr{A}}(C, -) : \mathscr{A} \to \mathsf{Ab}$ commutes with (small) coproducts.*

(ii) The category $\mathscr{A}$ is called quasi-elementary *if it is cocomplete and has a small generating subcategory $\mathscr{P} \subseteq \mathscr{A}$ of small projective objects.*

Definition 2.1.5. *[Kel24, Definition 2.97] Let $\mathcal{S}$ be a collection of morphisms in a cocomplete quasi-abelian category $\mathscr{A}$ stable under composition.*

(i) Let $\mathcal{I}$ be a filtered category. An object $C \in \mathscr{A}$ is called $(\mathcal{I}, \mathcal{S})$-tiny *if the functor $\mathrm{Hom}(C, -) : \mathscr{A} \to \mathsf{Ab}$ commutes with colimits of diagrams in $\mathrm{Fun}_{\mathcal{S}}(\mathcal{I}, \mathscr{A})$. Here $\mathrm{Fun}_{\mathcal{S}}(\mathcal{I}, \mathscr{A}) \subseteq \mathrm{Fun}(\mathcal{I}, \mathscr{A})$ denotes the sub-class of those functors which take morphisms in $\mathcal{I}$ into $\mathcal{S}$.*

(ii) The category $\mathscr{A}$ is called $(\mathcal{I}, \mathcal{S})$-elementary *if $\mathscr{A}$ is generated by a subcategory $\mathscr{P} \subseteq \mathscr{A}$ consisting of $(\mathcal{I}, \mathcal{S})$-tiny projective objects.*

(iii) An object $C \in \mathscr{A}$ is called $\mathcal{S}$-tiny *if the functor $\mathrm{Hom}(C, -) : \mathscr{A} \to \mathsf{Ab}$ commutes with colimits of diagrams in $\mathrm{Fun}_{\mathcal{S}}(\mathcal{I}, \mathscr{A})$, for any filtered category $\mathcal{I}$.*

(iv) The category $\mathscr{A}$ is called $\mathcal{S}$-elementary *if $\mathscr{A}$ is generated by a subcategory $\mathscr{P} \subseteq \mathscr{A}$ consisting of $\mathcal{S}$-tiny projective objects.*

In what follows we will often take $(\mathcal{I}, \mathcal{S}) := (\mathbf{N}, \mathrm{SplitMon})$, where SplitMon is the class of split monomorphisms in $\mathscr{A}$, or $\mathcal{S} := \mathrm{AdMon}$ to be the class of strict monomorphisms in $\mathscr{A}$, or $\mathcal{S} :=$ all. In each of these cases we say that $\mathscr{A}$ is $(\mathbf{N}, \mathrm{SplitMon})$-*elementary*, AdMon-*elementary*, and *elementary*, respectively. Of course, we have the following chain of implications:

$$\begin{array}{c} \text{elementary} \\ \Downarrow \\ \text{AdMon-elementary} \\ \Downarrow \\ \text{quasi-elementary} \\ \Downarrow \\ (\mathbf{N}, \mathrm{SplitMon})\text{-elementary} \\ \Downarrow \\ \text{enough projectives.} \end{array} \tag{2.1}$$

Notations 2.1.6. *Let $\mathscr{A}$ be an additive category and let $\mathrm{Ch}(\mathscr{A})$ denote the category of cochain complexes. In this thesis we always use* superscripts *to denote* cohomological *indexing convention and* subscripts *for* homological *indexing. These conventions are related by $M^i = M_{-i}$ for $i \in \mathbf{Z}$.*

Theorem 2.1.7. *[Kel24, Theorem 4.59, Theorem 4.65]*

(i) Let $\mathscr{A}$ be a quasi-abelian category with enough projectives. Then the projective model structure *on $\mathrm{Ch}^{\leqslant 0}(\mathscr{A})$ exists. The weak equivalences, fibrations and cofibrations may be described as follows:*

(W) A morphism is a weak equivalence if it is a strict quasi-isomorphism, i.e., its cone is strictly exact.

(F) *A morphism is a fibration if the its components are strict epimorphisms in positive degrees.*

(C) *A morphism is a cofibration if it is a degreewise strict monomorphism with degreewise projective cokernel.*

Further, this is a simplicial model structure.

(ii) *Assume that* $\mathscr{A}$ *is a* $(\mathbf{N}, \mathrm{SplitMon})$*-elementary quasi-abelian category. Then the* projective model structure *on* $\mathrm{Ch}(\mathscr{A})$ *exists. The weak equivalences and fibrations may be described as follows:*

(W) *A morphism is a weak equivalence if it is a strict quasi-isomorphism, i.e., its cone is strictly exact.*

(F) *A morphism is a fibration if it is a degreewise strict epimorphism.*

Further, this is a stable and simplicial model structure.

This permits us to make the following definition.

Definition 2.1.8. *Let* $\mathscr{A}$ *be a* $(\mathbf{N}, \mathrm{SplitMon})$*-elementary quasi-abelian category. The* derived ∞-category *of* $\mathscr{A}$ *is defined to be the underlying* ∞*-category of the simplicial model category* $\mathrm{Ch}(\mathscr{A})$*. That is, it is the* ∞*-categorical localization*

$$D(\mathscr{A}) := N(\mathrm{Ch}(\mathscr{A}))[W^{-1}]. \tag{2.2}$$

This is a stable ∞*-category.*

We recall [Sch99, §1.2.2] that $D(\mathscr{A})$ is equipped with two canonical t-structures. Of these, it is conventional to prefer the *left t-structure* which may be described as follows.[2]

Proposition 2.1.9. *[Sch99, §1.2.2] Let* $D^{\leqslant 0}(\mathscr{A})$ *(resp.* $D^{\geqslant 0}(\mathscr{A})$*) denote the full sub* ∞*-category of complexes which are strictly exact in positive (resp. negative) degrees. Then the pair*

$$(D^{\leqslant 0}(\mathscr{A}), D^{\geqslant 0}(\mathscr{A})) \tag{2.3}$$

determines a t-structure on $D(\mathscr{A})$*.*

The heart of this t-structure is called the *left heart* of $\mathscr{A}$ [Sch99, §1.2.3] and denoted by $LH(\mathscr{A})$. Consequently, we get cohomology functors

$$H^i : D(\mathscr{A}) \to LH(\mathscr{A}) \tag{2.4}$$

for each $i \in \mathbf{Z}$. The category $LH(\mathscr{A})$ admits the following very explicit description. Let $K(\mathscr{A})$ be the category with $\mathrm{Ob}(K(\mathscr{A})) = \mathrm{Ob}(\mathrm{Ch}(\mathscr{A}))$ and whose morphisms are chain-homotopy classes of morphisms in $\mathrm{Ch}(\mathscr{A})$.

Proposition 2.1.10. *[Sch99, Corollary 1.2.21] The category* $LH(\mathscr{A})$ *is equivalent to the full subcategory of* $K(\mathscr{A})$ *on two-term complexes*

$$0 \to M^{-1} \xrightarrow{d} M^0 \to 0 \tag{2.5}$$

[2] We recall that a t-structure on a stable ∞-category, is *by definition* given by a t-structure on its homotopy category.

with d a monomorphism, localized *at the class of morphisms $f : [M^{-1} \to M^0] \to [N^{-1} \to N^0]$ such that*

$$\begin{array}{ccc} M^{-1} & \longrightarrow & M^0 \\ {\scriptstyle f^{(-1)}}\downarrow & & \downarrow{\scriptstyle f^0} \\ N^{-1} & \longrightarrow & N^0 \end{array} \tag{2.6}$$

is a Cartesian and coCartesian square in $\mathscr{A}$.

With respect to this description, we obtain the following:

Proposition 2.1.11. *[Sch99, §1.2], see also [Bod21, §3.1].*

(i) The canonical functor $I : \mathscr{A} \to LH(\mathscr{A})$ is induced by the functor given by

$$M \mapsto [0 \to M]. \tag{2.7}$$

This is fully-faithful and admits a left adjoint $C : LH(\mathscr{A}) \to \mathscr{A}$ which is induced by the functor

$$[M^{-1} \xrightarrow{d} M^0] \mapsto \operatorname{coker} d. \tag{2.8}$$

on two-term complexes, so that $\mathscr{A}$ is a reflective subcategory of $LH(\mathscr{A})$.

(ii) The cohomology functor $H^i : D(\mathscr{A}) \to LH(\mathscr{A})$ is given on objects by

$$H^i : M^\bullet \mapsto [\operatorname{coker} \ker d^{i-1} \to \ker d^i]. \tag{2.9}$$

In particular, a complex $M^\bullet \in D(\mathscr{A})$ is strict (meaning that its differentials are strict morphisms) if and only if the cohomology objects $H^i M^\bullet \in LH(\mathscr{A})$ factor through (the essential image of) $\mathscr{A}$. A complex $M^\bullet$ is strictly exact if and only if $H^i M^\bullet = 0$ for all $i \in \mathbf{Z}$.

2.1.1 Banach rings and complete bornological modules

Remark 2.1.12 (Important remark)**.** *In this thesis we only consider* non-Archimedean *Banach algebras and* non-Archimedean *Banach modules. However, it seems quite likely, or perhaps even obvious, that much of the content of this thesis carries over to the Archimedean setting.*

Our conventions on Banach rings and modules follows Berkovich [Ber90, Chapter 1].

Definition 2.1.13. *(i) Let V be an abelian group. A (non-Archimedean)* seminorm *on V is a function $\|\cdot\| : V \to \mathbf{R}^{\geqslant 0}$ such that $\|0\| = 0$ and $\|v - w\| \leqslant \max\{\|v\|, \|w\|\}$ for all $v, w \in V$. It is called a* norm *if $\|v\| = 0$ implies $v = 0$. V is called* complete *if it is complete as a metric space. We write $\widehat{V}$ for the completion of a seminormed abelian group V. Seminorms $\|\cdot\|$ and $\|\cdot\|'$ are called* equivalent *if there exists $C, C' \in \mathbf{R}^{>0}$ such that $C\|\cdot\| \leqslant \|\cdot\|' \leqslant C'\|\cdot\|$.*

(ii) Let R be a ring. A seminorm *on R is a seminorm on the abelian group $(R, +)$ such that $\|1\| = 1$ and $\|fg\| \leqslant \|f\| \cdot \|g\|$ for all $f, g \in R$. It is called* multiplicative *if $\|fg\| = \|f\| \cdot \|g\|$ is satisfied for all $f, g \in R$. A* Banach ring *is a normed ring which is complete as a metric space.*

(iii) Let R be a seminormed ring. A seminormed R-module *is a R-module V equipped with a seminorm $\|\cdot\|_V$ such that there exists $C \in \mathbf{R}^{>0}$ such that $\|fv\|_V \leqslant C\|f\|_R\|v\|_V$ for all $f \in R$ and all $v \in V$. If R is a Banach ring, then such V is called a* Banach R-module *if it is complete as a metric space.*

(iv) A non-Archimedean field *is a field which is complete with respect to a multiplicative seminorm.*

(v) Let R be a Banach ring and let V, W be Banach R-modules. The completed tensor product $V\widehat{\otimes}_R W$ *is defined to be the completion of $V \otimes_R W$ with respect to the norm*

$$\|x\| := \inf\left\{\max_i \|v_i\|\|w_i\| : x = \sum v_i \otimes w_i\right\}. \tag{2.10}$$

The internal Hom*, written $\underline{\mathrm{Hom}}_R(V, W)$ is the R-module of bounded R-linear maps $\mathrm{Hom}_R(V, W)$ equipped with the* operator norm*.*

Definition 2.1.14. *Let K be a non-trivially valued non-Archimedean field with unit ball $o \subseteq K$. Let V be a K-vector space. A* bornology *on V is a collection of $\mathscr{B}$ of* bounded subsets *of V satifying the following properties:*

(i) If $B \in \mathscr{B}$ and $B' \subseteq B$ then $B' \in \mathscr{B}$;

(ii) If $v \in V$ then $\{v\} \in \mathscr{B}$;

(iii) $\mathscr{B}$ is closed under finite unions;

(iv) If $B \in \mathscr{B}$ and $r \in K$ then $rB \in \mathscr{B}$;

(v) If $B \in \mathscr{B}$ then the o-submodule $o \cdot B \in \mathscr{B}$.

The pair $V = (V, \mathscr{B})$ is called a (convex) bornological K-vector space. *A morphism $T : V \to W$ of K-vector spaces is called* bounded *if $T(B) \subseteq W$ is bounded for every bounded subset $B \subseteq V$. In this way we obtain the category Born_K of bornological K-vector spaces.*

Proposition 2.1.15. *[HN70, PS00a, BBB16]. Let K be a non-trivially valued non-Archimedean field with unit ball $o \subseteq K$.*

(i) The category Born_K is closed symmetric monoidal. The bornological tensor product *is defined to be $V \otimes_K W$ endowed with the bornology generated by the collection of $B \otimes_o B'$ for B, B' bounded o-submodules of V, W. The* internal Hom $\underline{\mathrm{Hom}}_K(V, W)$ *is the K-vector space $\mathrm{Hom}_K(V, W)$ of bounded linear maps equipped with the bornology generated by equibounded subsets.*

(ii) Born_K is a complete and cocomplete AdMon*-elementary quasi-abelian category.*

(iii) A generating family of AdMon*-tiny projective objects is given by $\{\coprod_S^{\leqslant 1} K\}_S$ for S ranging over (small) sets. Here $\coprod_S^{\leqslant 1} K$ is the normed K-vector space with underlying K-vector space $\coprod_S K$ and norm $\|(r_s)_s\| := \sup_s \|r_s\|$.*

(iv) A morphism $\varphi : V \to W$ is strict if and only if the subspace bornology on $\operatorname{im}\varphi$ coincides with the quotient bornology on $V/\ker\varphi$.

Every seminormed K-vector space acquires a bornology in an obvious way. Therefore we may make the following definition. The point is that the norm should not be the part of the data of a K-Banach space, only the bornology.

Definition 2.1.16. *Let K be a non-trivially valued non-Archimedean field. The category of* seminormed K-vector spaces*, (resp.* normed K-vector spaces*, resp.* K-Banach spaces*), is defined to be the full subcategory of Born_K on objects V whose bornology is induced by a seminorm (resp. a norm, resp. a norm making V into a Banach space). We denote these categories by $\mathsf{SNrm}_K, \mathsf{Nrm}_K$, and Ban_K, respectively.*

Definition 2.1.17. *Let K be a non-trivially valued non-Archimedean field and let $V \in \mathsf{Born}_K$. Given a bounded o-submodule $B \subseteq V$ we define $V_B := \operatorname{span}_K B \subseteq V$ equipped with the bornology defined by the gauge seminorm:*

$$\|x\|_B := \inf\{|\lambda| : x \in \lambda B\}. \tag{2.11}$$

Proposition 2.1.18. *[HN70, PS00a]. Let K be a non-trivially valued non-Archimedean field. There is an adjunction*

$$\text{diss} : \mathsf{Born}_K \leftrightarrows \mathsf{Ind}(\mathsf{SNrm}_K) : \text{colim} \tag{2.12}$$

in which the right adjoint diss : $V \mapsto$ "colim" V_B *is fully faithful. The essential image is given by the* essentially monomorphic Ind-objects*, i.e., those* Ind*-objects which are equivalent to* Ind*-systems of monomorphisms. Consequently there is an equivalence of categories*

$$\text{colim} : \mathsf{Ind}^m(\mathsf{SNrm}_K) \xrightarrow{\sim} \mathsf{Born}_K. \tag{2.13}$$

Definition 2.1.19. *[HN70, PS00a]. Let K be a non-trivially valued non-Archimedean field and let $V \in \mathsf{Born}_K$.*

(i) V is called separated *if, for every $B \in \mathscr{B}$ there exists a bounded o-submodule $B' \supseteq B$ such that the gauge seminorm on $V_{B'}$ is a norm. We let $\mathsf{SBorn}_K \subseteq \mathsf{Born}_K$ denote the full subcategory on separated bornological K-vector spaces.*

(ii) V is called complete *if, for every $B \in \mathscr{B}$ there exists a bounded o-submodule $B' \supseteq B$ such that $V_{B'}$ is a K-Banach space. We let $\mathsf{CBorn}_K \subseteq \mathsf{Born}_K$ denote the full subcategory on complete bornological K-vector spaces.*

Proposition 2.1.20. *[HN70, PS00a]. Let K be a non-trivially valued non-Archimedean field.*

(i) The inclusion $\mathsf{SBorn}_K \subseteq \mathsf{Born}_K$ admits a left adjoint $\text{sep} : \mathsf{Born}_K \to \mathsf{SBorn}_K$.

(ii) The category SBorn_K is closed symmetric monoidal. The tensor product is given by $\text{sep}(V \otimes_K W)$ *and the internal Hom is the same as in Born_K.*

(iii) SBorn_K is a complete and cocomplete AdMon*-elementary quasi-abelian category.*

(iv) A generating family of AdMon*-tiny projective objects is given by $\{\coprod_S^{\leqslant 1} K\}_S$ for S ranging over (small) sets.*

(v) A morphism $\varphi : V \to W$ is strict if and only if $\operatorname{im}\varphi$ is bornologically closed and the bornology on $\operatorname{im}\varphi$ coincides with the quotient bornology on $V/\ker\varphi$.

(vi) There is an adjunction

$$\text{diss} : \mathsf{SBorn}_K \leftrightarrows \mathsf{Ind}(\mathsf{Nrm}_K) : \text{colim} \tag{2.14}$$

in which the right adjoint diss : $V \mapsto$ "colim" $\text{sep}(V_B)$ *is fully-faithful and whose essential image is given by the the essentially monomorphic* Ind-*objects, so that there is an equivalence of categories*

$$\text{colim} : \mathsf{Ind}^m(\mathsf{Nrm}_K) \xrightarrow{\sim} \mathsf{SBorn}_K. \tag{2.15}$$

Definition 2.1.21. *Let K be a non-trivially valued non-Archimdean field and let V be a K-Banach space. Let S be a (small) set. We define the* space of V-valued zero sequences *to be*

$$c_0(S, V) := \{\phi : S \to V : \forall \varepsilon > 0, \exists \text{ at most finitely many } s \in S : \|\phi(s)\| > \varepsilon\}, \tag{2.16}$$

with the norm $\|\phi\| := \sup_{s \in S} \|\phi(s)\|$. When $V = K$ we will just write $c_0(S) := c_0(S, K)$.

Lemma 2.1.22. *Let K be a non-trivially valued non-Archimedean field and let $V \in \mathsf{CBorn}_K$.*

(i) There is a natural isomorphism

$$\text{Hom}_K(c_0(S), V) \cong \{\text{functions } f : S \to V : f(S) \subseteq V \text{ is bounded}\}. \tag{2.17}$$

(ii) For every S, S', there are natural isomorphisms

$$c_0(S)\widehat{\otimes}_K c_0(S') \cong c_0(S \times S') \cong c_0(S, c_0(S')), \tag{2.18}$$

of K-Banach spaces.

Proof. We omit the proof of (i). We only mention that (ii) can be proved using (i) together with currying and Yoneda's lemma. ☐

Proposition 2.1.23. *[HN70, PS00a] Let K be a non-trivially valued non-Archimedean field.*

(i) The inclusion $\mathsf{CBorn}_K \subseteq \mathsf{Born}_K$ admits a left adjoint $\widehat{(\cdot)} : \mathsf{Born}_K \to \mathsf{CBorn}_K$.

(ii) The category CBorn_K is closed symmetric monoidal. The tensor product is given by the completed tensor product

$$V\widehat{\otimes}_K W := \widehat{V \otimes_K W}, \tag{2.19}$$

and the internal Hom is the same as in Born_K.

(iii) CBorn_K is a complete and cocomplete AdMon-*elementary quasi-abelian category. A generating family of* AdMon-*tiny projective objects is given by $\{c_0(S)\}_S$ for S ranging over (small) sets. A morphism $\varphi : V \to W$ is strict if and only if $\operatorname{im} \varphi$ is bornologically closed and the bornology on $\operatorname{im} \varphi$ coincides with the quotient bornology on $V / \ker \varphi$.*

(iv) There is an adjunction

$$\mathrm{diss} : \mathsf{CBorn}_K \leftrightarrows \mathsf{Ind}(\mathsf{Ban}_K) : \mathrm{colim} \tag{2.20}$$

in which the right adjoint $\mathrm{diss} : V \mapsto \text{“colim”}(\widehat{V_B})$ *is fully-faithful and whose essential image is given by the the essentially monomorphic* Ind*-objects, so that there is an equivalence of categories*

$$\mathrm{colim} : \mathsf{Ind}^m(\mathsf{Ban}_K) \xrightarrow{\sim} \mathsf{CBorn}_K. \tag{2.21}$$

Corollary 2.1.24. *Let K be a non-trivially valued non-Archimedean field. In any of the categories Born_K, SBorn_K and CBorn_K, colimits of (essentially) monomorphic filtered systems are strongly exact.*

It is possible to give a more efficient “colimit presentation” for bornological K-vector spaces than the ones presented above. For this purpose we make the following Definition.

Definition 2.1.25. *Let K be a non-trivially valued non-Archimedean field and let $V \in \mathsf{Born}_K$.*

(i) We define a transitive, reflexive relation $\leqslant$ on the bounded o-submodules of V by $B \leqslant B'$ if there exists a bounded o-submodule $S \subseteq K$ such that $B \subseteq S \cdot B'$. If $B \leqslant B'$ we say that B' absorbs *B.*

(ii) We define an equivalence relation $\sim$ on the bounded o-submodules of V by $B \sim B'$ if $B \leqslant B'$ and $B' \leqslant B$.

(iii) We let $(\mathfrak{S}(V), \leqslant)$ denote the collection of $\sim$-equivalence classes viewed as a poset with the partial order induced by $\leqslant$.

Lemma 2.1.26. *Let K be a non-trivially valued non-Archimedean field.*

(i) If $V \in \mathsf{Born}_K$ then $\text{“colim”}_{[B]\in\mathfrak{S}(V)} V_B \in \mathsf{Ind}(\mathsf{SNrm}_K)$ is essentially monomorphic and the natural morphism

$$\operatorname*{colim}_{[B]\in\mathfrak{S}(V)} V_B \xrightarrow{\sim} V \tag{2.22}$$

is an isomorphism in Born_K.

(ii) If $V \in \mathsf{SBorn}_K$ then $\text{“colim”}_{[B]\in\mathfrak{S}(V)} \mathrm{sep}(V_B) \in \mathsf{Ind}(\mathsf{Nrm}_K)$ is essentially monomorphic and the natural morphism

$$\operatorname*{colim}_{[B]\in\mathfrak{S}(V)} \mathrm{sep}(V_B) \xrightarrow{\sim} V \tag{2.23}$$

is an isomorphism in SBorn_K.

(iii) If $V \in \mathsf{CBorn}_K$ then $\text{“colim”}_{[B]\in\mathfrak{S}(V)} \widehat{V_B} \in \mathsf{Ind}(\mathsf{Ban}_K)$ is essentially monomorphic and the natural morphism

$$\operatorname*{colim}_{[B]\in\mathfrak{S}(V)} \widehat{V_B} \xrightarrow{\sim} V \tag{2.24}$$

is an isomorphism in CBorn_K.

Proof. Given the previous Propositions, the only thing to note is that if $B \sim B'$ then there is an equality $V_B = V_{B'}$. □

Example 2.1.27. *An object $V \in \mathsf{Born}_K$ belongs to the full subcategory SNrm_K if and only if $\mathfrak{S}(V)$ has a terminal object.*

Definition 2.1.28. *[HN70] Let K be a non-trivially valued non-Archimedean field and let $V \in \mathsf{Born}_K$.*

(i) We say that V is of countable type *if it has a countable base for its bornology. Equivalently, the poset $\mathfrak{S}(V)$ has a countable cofinal subset.*

(ii) We say that V is metrizable *if the poset $\mathfrak{S}(V)$ is $\aleph_1$-filtered.*

The relation to locally-convex vector spaces is the following.

Proposition 2.1.29. *[HN70] Let K be a non-trivially valued non-Archimedean field. There is an adjunction*

$$(-)^t : \mathsf{Born}_K \leftrightarrows \mathsf{LCVS}_K : (-)^b \tag{2.25}$$

which is given as follows:

- *The functor $W \mapsto W^b$ endows a locally-convex K-vector space W with its* von Neumann bornology*: One has $W^b = W$ as K-vector spaces, and a subset $B \subseteq W^b$ is bounded if for every lattice $L \subseteq V$ there exists $\lambda \in K$ such that $B \subseteq \lambda L$.*
- Dually*, the functor $V \mapsto V^t$ endows V with the* topology of bornivorous subsets*: One has $V^t = V$ as K-vector spaces, and an o-submodule $L \subseteq V^t$ is an open neighbourhood of* 0 *if for every bounded subset $B \subseteq V$ there exists $\lambda \in K$ such that $B \subseteq \lambda L$.*

Lemma 2.1.30. *[HN70] Let K be a non-trivially valued non-Archimedean field.*

(i) If $W \in \mathsf{LCVS}_K$ is metrizable, then the counit morphism $W^{bt} \to W$ is an isomorphism. In particular we obtain a fully-faithful functor from the full subcategory of metrizable objects to Born_K.

(ii) If $W \in \mathsf{LCVS}_K$ is complete and metrizable, then W^b is complete as a bornological space and metrizable in the sense of Definition 2.1.28. In particular we obtain a fully-faithful functor $(-)^b : \mathsf{Fr}_K \hookrightarrow \mathsf{CBorn}_K$ from the category Fr_K of K-Fréchet spaces.

Proof. We only prove the metrizability part in (ii), the rest of the assertions being well-known. This appears already in [HN70, p.200] where it is attributed to Mackey. Let $\{B_n\}_n$ be a countable collection of bounded o-submodules of V and let $\{L_n\}_n$ be a (decreasing) fundamental system of lattices defining the Fréchet topology. By definition of the von Neumann bornology, for each n there exists $\lambda_n \in K^\times$ such that $\lambda_n B_n \subseteq L_n$. Set $B' := \sum_n \lambda_n B_n$. Obviously, $B_n \leqslant B'$ for each n. Further, for each $N \geqslant 0$ one has $B' \subseteq L_N + \sum_{n=1}^N \lambda_n B_n$, which implies that B' is von Neumann bounded. □

Definition 2.1.31. *Let K be a non-trivially valued non-Archimdean field. An object $V \in \mathsf{CBorn}_K$ is called* conuclear *if for all K-Banach spaces W, the canonical morphism*

$$\underline{\mathrm{Hom}}_K(V, K)\widehat{\otimes}_K W \to \underline{\mathrm{Hom}}_K(V, W) \tag{2.26}$$

is an isomorphism.

Example 2.1.32. *Let K be a non-trivially valued non-Archimedean field. Let $\varpi \in K$ with $0 < |\varpi| < 1$. Then*

$$K\langle x/\varpi^{\infty}\rangle := \operatorname*{colim}_n K\langle x/\varpi^n\rangle = \Big\{\sum_{n=0}^{\infty} a_n x^n : |a_n||\varpi|^{-nk} \xrightarrow{n\to\infty} 0 \textit{ for some } k \in \mathbf{Z}\Big\}.$$

is a conuclear object of CBorn_K.

Remark 2.1.33. *It is natural to ask how the definition of CBorn_K generalizes when we replace K by a general Banach ring R. It seems like the good definition is $\mathsf{CBorn}_R := \mathsf{Ind}^m\mathsf{Ban}_R$. By Proposition 2.1.23(iv) this is consistent with our previous definition in the case when $R = K$ is a non-trivially valued non-Archimedean field.*

2.1.2 Universal property of the derived category

In this section we assume that

- $\star$ $\mathscr{A}$ is a quasi-elementary quasi-abelian category.

We fix a small generating set of small projective objects $\mathscr{P} \subseteq \mathscr{A}$, and we assume that $\mathscr{P}$ is closed under finite products in $\mathscr{A}$ (there is no harm in this). We always view $\mathscr{P}$ as a full subcategory of $\mathscr{A}$. There is a functor

$$\mathscr{A} \to \mathrm{Fun}^{\Pi}(\mathscr{P}^{\mathsf{op}}, \mathsf{Set}) \tag{2.27}$$

which sends $A \mapsto h_A := \mathrm{Hom}(-, A)$. (Here and elsewhere, Fun^{Π} denotes the *finite* product-preserving functors). Passing to simplicial objects, we obtain

$$s\mathscr{A} \to \mathrm{Fun}^{\Pi}(\mathscr{P}^{\mathsf{op}}, s\mathsf{Set}) \tag{2.28}$$

sending $A_\bullet \mapsto h_{A_\bullet} := \mathrm{Hom}(-, A_\bullet)$.

Proposition 2.1.34. *[Qui67, §II.4] With notations as above. We may define a model structure (called the* standard model structure*) on $s\mathscr{A}$ as follows: A morphism $A_\bullet \to B_\bullet$ is a weak equivalence (resp. fibration) if $\mathrm{Hom}(P, A_\bullet) \to \mathrm{Hom}(P, B_\bullet)$ is a weak homotopy equivalence (resp. Kan fibration) in $s\mathsf{Set}$, for all $P \in \mathscr{P}$.*

Proposition 2.1.35 (Quillen, Bergner, Lurie)**.** *(i) We may define a model structure on $\mathrm{Fun}^{\Pi}(\mathscr{P}^{\mathsf{op}}, s\mathsf{Set})$ as follows: A morphism $\alpha : F \to F'$ is a weak equivalence (resp. fibration) if $\alpha_P : F(P) \to F'(P)$ is a weak homotopy equivalence (resp. Kan fibration) in $s\mathsf{Set}$, for all $P \in \mathscr{P}$.*

(ii) This model structure presents the ∞-category $\mathrm{Fun}^{\Pi}(N(\mathscr{P})^{\mathsf{op}}, \infty\mathsf{Grpd}) = \mathrm{sInd}(N(\mathscr{P}))$.

(iii) By transport of structure via (2.28)*, the model structure of (i) gives the standard model structure on $s\mathscr{A}$.*

Proof. (i) This is [Lur09b, Proposition 5.5.9.1]. (ii): This is [Lur09b, Corollary 5.5.9.3]. (iii): Obvious. ☐

Proposition 2.1.36. *Under the equivalence of categories*

$$N : s\mathscr{A} \simeq \mathrm{Ch}^{\leqslant 0}(\mathscr{A}) : \Gamma \tag{2.29}$$

furnished by the Dold-Kan correspondence, the model structure on $s\mathscr{A}$ induces the model structure on $\mathrm{Ch}^{\leqslant 0}(\mathscr{A})$ and vice-versa.

Proof. The functors N and Γ furnish an equivalence of categories by, for instance, [Kel24, Corollary 4.73]. For each $P \in \mathscr{P}$ we have a commutative diagram

$$\begin{array}{ccc} s\mathscr{A} & \xrightarrow{N} & \mathrm{Ch}^{\leqslant 0}(\mathscr{A}) \\ {\scriptstyle \mathrm{Hom}(P,-)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(P,-)} \\ s\mathsf{Ab} & \xrightarrow{N} & \mathrm{Ch}^{\leqslant 0}(\mathsf{Ab}) \end{array} \tag{2.30}$$

By definition, $f : X_\bullet \to Y_\bullet$ is a fibration if and only if $\mathrm{Hom}(P, X_\bullet) \to \mathrm{Hom}(P, Y_\bullet)$ is a fibration for all $P \in \mathscr{P}$. By (the generalization to abelian groups of) [Qui67, Proposition 2.3.1], together with commutativity of the above square, this holds if and only if $\mathrm{Hom}(P, N(X_\bullet)) \to \mathrm{Hom}(P, N(Y_\bullet))$ is an epimorphism (of chain complexes) in positive degrees. By [Qui67, Proposition 2.4.2] this implies that $N(X_\bullet) \to N(Y_\bullet)$ is a degreewise strict epimorphism (we recall that in additive categories, the notion of strict and effective epimorphism coincide). Hence f is a fibration if and only if $N(f)$ is.

Similarly, a morphism $f : X_\bullet \to Y_\bullet$ is a weak equivalence if and only if $\mathrm{Hom}(P, N(X_\bullet)) \to \mathrm{Hom}(P, N(Y_\bullet))$ is a quasi-isomorphism of chain complexes, for each $P \in \mathscr{P}$. Looking at the cone of $N(X_\bullet) \to N(Y_\bullet)$ and applying [Kel24, Corollary 2.95] this holds if and only if $N(X_\bullet) \to N(Y_\bullet)$ is a strict quasi-isomorphism. Hence f is a weak equivalence if and only if $N(f)$ is. □

Corollary 2.1.37. *The model structure on $s\mathscr{A}$ may be equivalently described as follows: A morphism $A_\bullet \to B_\bullet$ is a weak equivalence (resp. fibration, resp. cofibration), if and only if it is a strict weak homotopy equivalence (resp. strict epimorphism in positive degrees, resp. degreewise strict monomorphism with degreewise projective cokernel).*

Proof. It should be possible to argue directly, looking at the proof of [Qui67, §II.4]. In any case, we can use Proposition 2.1.36, whence the claims about weak equivalences and fibrations are clear. The statement about cofibrations can be deduced from the corresponding statement for $\mathrm{Ch}^{\leqslant 0}(\mathscr{A})$, which is contained in [Kel24, Theorem 4.65]. □

Theorem 2.1.38. *With notation as above. The functor* (2.28) *induces an equivalence of ∞-categories*

$$N(s\mathscr{A})[W^{-1}] \xrightarrow{\sim} \mathrm{sInd}(N(\mathscr{P})). \tag{2.31}$$

Proof. The following argument is quite similar to the proof of [Sch99, Proposition 2.1.14]. Let us write $\mathscr{A}' := \mathrm{Fun}^{\Pi}(\mathscr{P}^{\mathsf{op}}, \mathsf{Set})$. This is an abelian category (in fact it is equal to the left heart of $\mathscr{A}$). The image of $\mathscr{P}$ under the Yoneda embedding gives a generating family of small (strongly) projective objects for $\mathscr{A}'$. Hence we may consider the standard model structure on $s\mathscr{A}'$. It is clear that there is an equivalence of categories

$$s\mathscr{A}' = \mathrm{Fun}^{\Pi}(\mathscr{P}^{\mathsf{op}}, s\mathsf{Set}) \tag{2.32}$$

which identifies this model structure with the one from Proposition 2.1.35(i). Let $\mathscr{L}$ be the category obtained from $\mathscr{P}$ by freely adjoining (small) direct sums. By the smallness assumption on objects of $\mathscr{P}$, both functors $\mathscr{L} \to \mathscr{A}$ and $\mathscr{L} \to \mathscr{A}'$ are fully-faithful, factor through the projective objects, and induce fully-faithful functors $s\mathscr{L} \to s\mathscr{A}$ and $s\mathscr{L} \to s\mathscr{A}'$. By Corollary 2.1.37, the image of $s\mathscr{L}$ in $s\mathscr{A}$ or $s\mathscr{A}'$ consists of fibrantcofibrant objects. We thusly obtain a diagram

$$\begin{array}{ccc} N(s\mathscr{A})[W^{-1}] & \longrightarrow & N(s\mathscr{A}')[W^{-1}] \\ {\scriptstyle\simeq}\nwarrow & & \nearrow{\scriptstyle\simeq} \\ & N(s\mathscr{L})[H^{-1}] & \end{array} \tag{2.33}$$

in which $[H^{-1}]$ denotes the localization at simplicial homotopy equivalences. This completes the proof. □

Corollary 2.1.39 (Universal property of $D^{\leqslant 0}(\mathscr{A})$)**.** *(i) There is an equivalence*

$$D^{\leqslant 0}(\mathscr{A}) \simeq \mathrm{Fun}^{\Pi}(\mathscr{P}^{\mathsf{op}}, \infty\mathsf{Grpd}) = \mathrm{sInd}(N(\mathscr{P})). \tag{2.34}$$

(ii) Let $\mathscr{D}$ be any cocomplete ∞-category. Composition with $N(\mathscr{P}) \to D^{\leqslant 0}(\mathscr{A})$ induces an equivalence of ∞-categories

$$\mathrm{Fun}^{L}(D^{\leqslant 0}(\mathscr{A}), \mathscr{D}) \xrightarrow{\sim} \mathrm{Fun}^{\Pi}(N(\mathscr{P}), \mathscr{D}), \tag{2.35}$$

where Fun^L denotes the full subcategory spanned by colimit-preserving functors.

Proof. (i): Combine Proposition 2.1.35 and Proposition 2.1.36. (ii): Combine (i) and [Lur09b, Corollary 5.5.8.15(c)]. □

Definition 2.1.40. *[Lur17, Definition 5.5.7.1]*

(i) Let $\mathscr{C}$ be an ∞-category admitting filtered colimits. An object $C \in \mathscr{C}$ is called compact *if $\mathrm{Hom}(C, -)$ commutes with filtered colimits. We let $\mathscr{C}^{\omega} \subseteq \mathscr{C}$ be the full subcategory spanned by compact objects.*

(ii) An ∞-category $\mathscr{C}$ is called compactly generated *if there exists a small category $\mathscr{C}_0$ admitting finite colimits, and an equivalence of ∞-categories $\mathrm{Ind}(\mathscr{C}_0) \simeq \mathscr{C}$.*

Proposition 2.1.41. *[Lur09b, §5.5.7] Let $\mathscr{C}$ be a compactly generated ∞-category. Then the full subcategory $\mathscr{C}^{\omega} \subseteq \mathscr{C}$ is essentially small, admits finite colimits, and the inclusion induces an equivalence of ∞-categories $\mathrm{Ind}(\mathscr{C}^{\omega}) \xrightarrow{\sim} \mathscr{C}$.*

Lemma 2.1.42. *Let $\mathscr{C}$ be a compactly generated ∞-category. Then for any regular cardinal κ, one has that κ-small limits commute with κ-filtered colimits.*

Proof. This is well-known, but we include the proof for completeness. Let $p : I \times J \to \mathscr{C}$ be a diagram where I is κ-filtered and J is κ-small. We need to check that the canonical morphism

$$\operatorname*{colim}_I \lim_J p \to \lim_J \operatorname*{colim}_I p \tag{2.36}$$

is an equivalence. This can be checked after applying $\mathrm{Map}(C, -)$ for each $C \in \mathscr{C}^{\omega}$, reducing the proof of the Lemma to the case when $\mathscr{C} = \infty\mathsf{Grpd}$, which is [Lur09b, Proposition 5.3.3.3]. □

Lemma 2.1.43. *Let $F : \mathscr{C} \leftrightarrows \mathscr{D} : G$ be an adjunction between compactly generated ∞-categories. Then the right adjoint G preserves filtered colimits if and only if the left adjoint F preserves compact objects.*

Proof. This is again well-known, but we include the proof for completeness. The *only if* direction is clear by adjunction. For the *if* direction, suppose F preserves compact objects. Let $p : I \to \mathscr{D}$ be a filtered diagram. We need to check that the canonical morphism

$$\varphi : \operatorname{colim}_I G \circ p \to G \operatorname{colim}_I p \tag{2.37}$$

is an equivalence in $\mathscr{C}$. This can be checked after applying $\operatorname{Map}(C, -)$ for each $C \in \mathscr{C}^\omega$. For such C then $\operatorname{Map}(C, \varphi)$ is seen to be an equivalence, by adjunction and the fact that F preserves compact objects. □

Proposition 2.1.44 (Compact generation of $D^{\leqslant 0}(\mathscr{A})$)**.** *(i) The category $D^{\leqslant 0}(\mathscr{A})$ is compactly generated.*

(ii) Let $j : N(\mathscr{P}) \to D^{\leqslant 0}(\mathscr{A})$ be the inclusion. An object $C \in D^{\leqslant 0}(\mathscr{A})$ is compact if and only if the following holds: There exists a finite diagram $p : K \to \mathscr{P}$ such that C is a retract of $\operatorname{colim} j \circ p$.

Proof. By [Lur09b, Proposition 5.3.5.12], the ∞-category $\operatorname{PSh}(N(\mathscr{P}))$ is compactly generated and the compact objects admit precisely the description as in (ii), c.f. [Lur09b, Proposition 5.3.4.17]. Now, $\operatorname{sInd}(N(\mathscr{P}))$ is a localization of $\operatorname{PSh}(N(\mathscr{P}))$, and the inclusion $\operatorname{sInd}(N(\mathscr{P})) \hookrightarrow \operatorname{PSh}(N(\mathscr{P}))$ preserves sifted colimits[3], in particular filtered colimits. This implies that the left adjoint (the localization) preserves compact objects, giving both (i) and (ii). □

Remark 2.1.45. *Because compact objects are stable under finite colimits and retracts, we arrive at the following alternative description of $D^{\leqslant 0}(\mathscr{A})^\omega$: it is the full subcategory generated under cones, suspensions and retracts by $N(\mathscr{P}) \subseteq D^{\leqslant 0}(\mathscr{A})$. One might call these "connective $\mathscr{P}$-perfect complexes".*

Example 2.1.46. *Let K be a non-trivially valued non-Archimdedean field, let $\mathscr{A} = \mathsf{CBorn}_K$ and $\mathscr{P} = \{c_0(S)\}_S$ for S ranging over (small) sets. We obtain*

$$D^{\leqslant 0}(\mathsf{CBorn}_K) \simeq \operatorname{sInd}(N(\{c_0(S)\}_S)). \tag{2.38}$$

Definition 2.1.47. *Let $\mathscr{C}$ be a pointed ∞-category with finite limits. We let $\operatorname{Sp}(\mathscr{C})$ denote the* category of spectrum objects *of $\mathscr{C}$. It is the limit*

$$\operatorname{Sp}(\mathscr{C}) := \lim \left(\mathscr{C} \xleftarrow{\Omega} \mathscr{C} \xleftarrow{\Omega} \mathscr{C} \leftarrow \cdots \right) \tag{2.39}$$

computed in Cat_∞. The category Sp of spectra *is defined as $\mathsf{Sp} := \operatorname{Sp}(\infty\mathsf{Grpd}_*)$ where $\infty\mathsf{Grpd}_* := \infty\mathsf{Grpd}_{*/}$ denotes* pointed ∞*-groupoids.*

Lemma 2.1.48. *(i) The t-structure on $D(\mathscr{A})$ is both left and right complete;*

(ii) There are t-exact equivalences of ∞-categories

$$D(\mathscr{A}) \simeq \operatorname{Sp}(D^{\leqslant 0}(\mathscr{A})) \simeq \operatorname{Sp}(\operatorname{sInd}(N(\mathscr{P}))). \tag{2.40}$$

(iii) There is a t-exact equivalence of ∞-categories

$$\operatorname{Sp}(\operatorname{sInd}(N(\mathscr{P}))) \simeq \operatorname{Fun}^\Pi(N(\mathscr{P})^{\mathsf{op}}, \mathsf{Sp}). \tag{2.41}$$

[3]This is obvious since sInd is formed by freely adjoining sifted colimits, whereas PSh is formed by freely adjoining all colimits.

(iv) There is an equivalence of ∞-categories

$$D^{\leqslant 0}(\mathscr{A}) \otimes \mathsf{Sp} \simeq D(\mathscr{A}), \tag{2.42}$$

where the tensor product on the left is the Lurie tensor product on Pr^L.

Proof. (i): Our working assumption that $\mathscr{A}$ is quasi-elementary implies that products in $\mathscr{A}$ are exact and coproducts in $\mathscr{A}$ are (strongly) exact, c.f [Sch99, Proposition 2.1.15]. Hence both products and coproducts are t-exact in $D(\mathscr{A})$. From this it follows by [Lur17, Proposition 1.2.1.19] and its dual that $D(\mathscr{A})$ is left and right t-complete.

(ii): The equivalence $D(\mathscr{A}) \simeq \mathrm{Sp}(D^{\leqslant 0}(\mathscr{A}))$ is an immediate consequence of right-completeness of the t-structure, established in (i).

(iii): This is essentially [Lur11, Remark 1.2], but let us reproduce the proof here for convenience. Because the endofunctor Ω of $\infty\mathsf{Grpd}_*$ commutes with (finite) products, we can regard Sp as the limit of the tower

$$\infty\mathsf{Grpd}_* \xleftarrow{\Omega} \infty\mathsf{Grpd}_* \xleftarrow{\Omega} \infty\mathsf{Grpd}_* \xleftarrow{\Omega} \dots \tag{2.43}$$

computed in the ∞-category Cat_∞^Π of ∞-categories admitting finite products, with finite-product preserving functors. Consequently we obtain an equivalence

$$\mathrm{Fun}^\Pi(N(\mathscr{P})^{\mathsf{op}}, \mathsf{Sp}) \simeq \lim_\Omega \mathrm{Fun}^\Pi(N(\mathscr{P})^{\mathsf{op}}, \infty\mathsf{Grpd}_*) \tag{2.44}$$

and we note that there is a canonical equivalence

$$\mathrm{Fun}^\Pi(N(\mathscr{P})^{\mathsf{op}}, \infty\mathsf{Grpd}_*) \simeq \mathrm{Fun}^\Pi(N(\mathscr{P})^{\mathsf{op}}, \infty\mathsf{Grpd})_{*/} \tag{2.45}$$

Hence the conclusion will follow if we can show that $\mathrm{sInd}(N(\mathscr{P}))$ was already pointed. This follows from the fact that the Yoneda embedding $j : N(\mathscr{P}) \hookrightarrow \mathrm{sInd}(N(\mathscr{P}))$ preserves finite coproducts and all limits [Lur09b, Proposition 5.5.8.10], and $N(\mathscr{P})$ has a zero object.

(iv): Follows from (i), c.f. [Lur17, Example 4.8.1.23] (note that $D^{\leqslant 0}(\mathscr{A})$ is already pointed). ☐

Proposition 2.1.49 (Compact generation of $D(\mathscr{A})$). *(i) The ∞-category $D(\mathscr{A})$ is compactly generated.*

(ii) Let $j : N(\mathscr{P}) \hookrightarrow D(\mathscr{A})$ be the inclusion of $N(\mathscr{P})$ in degree 0. An object $C \in D(\mathscr{A})$ is compact if and only if the following holds: There exists $n \geqslant 0$ and a finite diagram $p : K \to \mathscr{P}$ such that C is a retract of $\operatorname{colim} \Omega^n j \circ p$.

Proof. Let us temporarily set $\mathscr{C} := D^{\leqslant 0}(\mathscr{A})$. By Proposition 2.1.44, $\mathscr{C}$ is compactly generated, so by Lemma 2.1.42 finite limits commute with filtered colimits in $\mathscr{C}$. In particular the loops functor $\Omega : \mathscr{C} \to \mathscr{C}$ commutes with filtered colimits. Therefore we can view

$$\mathscr{C} \xleftarrow{\Omega} \mathscr{C} \xleftarrow{\Omega} \mathscr{C} \xleftarrow{\Omega} \cdots \tag{2.46}$$

as a diagram in the ∞-category Pr^R_ω of compactly generated ∞-categories, with filtered-colimit preserving, right adjoint functors [Lur09b, Definition 5.5.7.5]. Now [Lur09b, Proposition 5.5.7.6] says that Pr^R_ω is closed under limits in Cat_∞, giving (i).

(ii): We examine the proof of *loc. cit.*. Let $\Omega^{\infty - n} : \mathrm{Sp}(\mathscr{C}) \to \mathscr{C}$ be the projection to the n^{th} component, which by (i) is a morphism in Pr^R_ω. Hence the left adjoint $\Sigma^{\infty - n} : \mathscr{C} \to \mathrm{Sp}(\mathscr{C})$ preserves compact objects. Together with Proposition 2.1.44 this implies that all objects as in (ii) are indeed compact. Moreover, [Lur09b, Lemma 6.3.3.6] implies that the identity functor on $\mathrm{Sp}(\mathscr{C})$ can be written as

$$\mathrm{id} \simeq \operatorname*{colim}_n \Sigma^{\infty-n}\Omega^{\infty-n}, \tag{2.47}$$

so $\mathrm{Sp}(\mathscr{C})$ is generated under filtered colimits by the essential images of the $\Sigma^{\infty-n}$. Since $\mathscr{C}$ is generated under filtered colimits by objects as in Proposition 2.1.44(ii), we see that objects as in (ii) generate $\mathrm{Sp}(\mathscr{C})$ under filtered colimits. A retract argument then shows that every compact object has the form as in (ii). ☐

Proposition 2.1.50 (Universal property of $D(\mathscr{A})$)**.** *(i) Let $\mathscr{D}$ be any stable presentable ∞-category. There is an equivalence of ∞-categories*

$$\mathrm{Fun}^L(D(\mathscr{A}), \mathscr{D}) \simeq \mathrm{Fun}^{\amalg}(N(\mathscr{P}), \mathscr{D}). \tag{2.48}$$

where Fun^L denotes the colimit-preserving functors and $\mathrm{Fun}^{\amalg}$ denotes the finite-coproduct preserving functors.

(ii) Let $\mathscr{D}$ be any stable presentable ∞-category with t-structure $(\mathscr{D}^{\leqslant 0}, \mathscr{D}^{\geqslant 0})$. There is an equivalence of ∞-categories

$$\mathrm{Fun}'(D(\mathscr{A}), \mathscr{D}) \simeq \mathrm{Fun}^{\amalg}(N(\mathscr{P}), \mathscr{D}^{\leqslant 0}). \tag{2.49}$$

where Fun' denotes the colimit-preserving and right t-exact functors.

Proof. (i): By Proposition 2.1.50 we have $D^{\leqslant 0}(\mathscr{A}) \otimes \mathsf{Sp} \simeq D(\mathscr{A})$ computed in the symmetric monoidal ∞-category Pr^L. Now $\mathrm{Fun}^L(-,-)$ is the internal Hom in Pr^L. Hence

$$\mathrm{Fun}^L(D^{\leqslant 0}(\mathscr{A}) \otimes \mathsf{Sp}, \mathscr{D}) \simeq \mathrm{Fun}^L(D^{\leqslant 0}(\mathscr{A}), \mathrm{Fun}^L(\mathsf{Sp}, \mathscr{D})), \tag{2.50}$$

but since $\mathscr{D}$ is stable and presentable we have $\mathrm{Fun}^L(\mathsf{Sp}, \mathscr{D}) \simeq \mathscr{D}$, proving (i). (ii): Clear from (i). ☐

Remark 2.1.51. *As in Remark 2.1.45, we arrive at the following alternative description of $D(\mathscr{A})^\omega$: it is the full subcategory generated under cones, shifts and retracts by $N(\mathscr{P}) \subseteq D(\mathscr{A})$. One might call these "$\mathscr{P}$-perfect complexes".*

2.1.3 Monoidal structure on $D(\mathscr{A})$

In this section we continue with the same assumptions as in §2.1.2 but further assume that:

- ⋆ $\mathscr{A}$ is endowed with a closed symmetric monoidal structure $(\mathscr{A}, \otimes, \underline{\mathrm{Hom}})$;
- ⋆ the monoidal structure on $\mathscr{A}$ restricts to $\mathscr{P}$;
- ⋆ every object of $\mathscr{P}$ is flat.

Example 2.1.52. *Let K be a non-trivially valued non-Archimedean field, let*

$$\mathscr{A} = (\mathsf{CBorn}_K, \widehat{\otimes}_K, \underline{\mathrm{Hom}}_K), \tag{2.51}$$

and $\mathscr{P} = \{c_0(S)\}_S$ for S ranging over (small) sets. There are canonical isomorphisms

$$c_0(S)\widehat{\otimes}_K c_0(S') \xrightarrow{\sim} c_0(S \times S'), \tag{2.52}$$

so that $\mathscr{P} \subseteq \mathscr{A}$ satisfies all the above assumptions.

Kelly has proved the following (the same result also holds for unbounded complexes, but for this section we only need the result in the connective case):

Theorem 2.1.53. *[Kel24, Theorem 4.69] Under the above assumptions. The projective model structure on* $\mathrm{Ch}^{\leqslant 0}(\mathscr{A})$ *is monoidal, i.e.,* $(\mathrm{Ch}^{\leqslant 0}(\mathscr{A}), \otimes)$ *is a monoidal model category.*

As a consequence of the dictionary between model categories and ∞-categories we obtain the following:

Corollary 2.1.54. *The* ∞*-category* $D^{\leqslant 0}(\mathscr{A})$ *is presentably symmetric monoidal, when endowed with the derived tensor product.*

Theorem 2.1.55. *There exists a symmetric monoidal structure* $\otimes_{\mathrm{Day}}$ *(called* Day convolution*) on* $\mathrm{sInd}(N(\mathscr{P}))$ *which is characterised up to equivalence by the following properties:*

(i) The Yoneda embedding $j : N(\mathscr{P}) \hookrightarrow \mathrm{sInd}(N(\mathscr{P}))$ *extends to a symmetric monoidal functor;*

(ii) $\otimes_{\mathrm{Day}}$ *commutes with colimits separately in each variable.*

Proof. The Theorem with *sifted colimits* in place of *all colimits* in (ii) is [Lur17, Proposition 4.8.1.10], taking $\mathcal{K} = \emptyset$ and $\mathcal{K}'$ to be the collection of sifted simplicial sets, in the notations of *loc. cit.*. To get the full statement of (ii), one argues *mutandis mutatis* as in [Lur17, Proposition 4.8.1.14], replacing the use of [Lur09b, Proposition 5.5.1.9] with [Lur09b, Remark 5.5.8.16(3)]. □

Corollary 2.1.56. *(i) The equivalence of Corollary 2.1.39(i) upgrades to an equivalence of presentably symmetric monoidal* ∞*-categories:*

$$(\mathrm{sInd}(N(\mathscr{P})), \otimes_{\mathrm{Day}}) \xrightarrow{\sim} (D^{\leqslant 0}(\mathscr{A}), \otimes^{\mathbf{L}}). \tag{2.53}$$

(ii) Let $\mathscr{D}$ *be any symmetric monoidal* ∞*-category such that* $\mathscr{D}$ *is cocomplete and the tensor product* $\mathscr{D} \times \mathscr{D} \to \mathscr{D}$ *preserves colimits separately in each variable. Then there is an equivalence of* ∞*-categories*

$$\mathrm{Fun}^{L,\otimes}(D^{\leqslant 0}(\mathscr{A}), \mathscr{D}) \simeq \mathrm{Fun}^{\Pi,\otimes}(N(\mathscr{P}), \mathscr{D}), \tag{2.54}$$

where $\mathrm{Fun}^{L,\otimes}$ *(resp.* $\mathrm{Fun}^{\Pi,\otimes}$*) denotes colimit-preserving (resp. finite product preserving) symmetric monoidal functors.*

Proof. (i): Both tensor products preserve colimits (separately in each variable) and restrict to the symmetric monoidal structure on $N(\mathscr{P})$. (ii): Follows from (i) using [Lur09b, Remark 5.5.8.16(3)]. □

Proposition 2.1.57. *There is a unique (up to equivalence) symmetric monoidal structure* $\otimes^{\mathbf{L}}$ *on* $D(\mathscr{A})$ *with the following properties:*

(i) The inclusion $D^{\leqslant 0}(\mathscr{A}) \hookrightarrow D(\mathscr{A})$ *extends to a symmetric monoidal functor;*

(ii) $\otimes^{\mathbf{L}}$ *commutes with colimits separately in each variable. In particular* $D(\mathscr{A})$ *is* presentably symmetric monoidal.

Proof. This follows immediately by interpreting the formula

$$D^{\leqslant 0}(\mathscr{A}) \otimes \mathsf{Sp} \simeq D(\mathscr{A}) \tag{2.55}$$

coming from Proposition 2.1.50(iv), as a tensor product of *commutative algebra objects* in Pr^L. ☐

Because $D(\mathscr{A})$ is *presentably symmetric monoidal*, it is in particular closed symmetric monoidal. We write

$$R\underline{\mathrm{Hom}}_{\mathscr{A}}(-,-) : D(\mathscr{A})^{\mathsf{op}} \times D(\mathscr{A}) \to D(\mathscr{A}) \tag{2.56}$$

for the internal Hom-bifunctor on $D(\mathscr{A})$.

Lemma 2.1.58. *(i) If $C^\bullet, C^{\prime,\bullet}$ are compact objects of $D(\mathscr{A})$, then so is $C^\bullet \otimes^{\mathbf{L}} C^{\prime,\bullet}$.*

(ii) If $C^\bullet \in D(\mathscr{A})$ is compact then the functor $R\underline{\mathrm{Hom}}_{\mathscr{A}}(C^\bullet, -)$ commutes with filtered colimits.

Proof. (i): By Proposition 2.1.57, the derived tensor product restricts to the ordinary tensor product on $N(\mathscr{P})$, and commutes with colimits (in particular suspensions and hence loops since $\mathscr{D}$ is stable) separately in each variable. Therefore the claim follows from the explicit description of Proposition 2.1.49(ii). (ii): By Lemma 2.1.43, this is immediate from (i). ☐

Remark 2.1.59. *Using Proposition 2.1.57 together the model stucture on $\mathrm{Ch}(\mathscr{A})$ and the formalism of bounded resolving classes (c.f. [Kel24, §2.3.2], [Spa88]) gives a way to calculate derived tensor products:*

- ⋆ *If $M^\bullet, N^\bullet \in \mathrm{Ch}^-(\mathscr{A})$, then one can find a degreewise projective complex $P^\bullet$ with an acyclic fibration $P^\bullet \to M^\bullet$. Then $P^\bullet \otimes N^\bullet \xrightarrow{\sim} M^\bullet \otimes^{\mathbf{L}} N^\bullet$ in $D(\mathscr{A})$.*
- ⋆ *If $M^\bullet, N^\bullet \in D(\mathscr{A})$, then there exists a direct system $\{P_n^\bullet\}_{n \geqslant -1}$ of bounded-above degreewise projective complexes, with a system of acyclic fibrations $P_n^\bullet \to \tau^{\leqslant n} M^\bullet$. Put $P^\bullet := \mathrm{colim}_{n \geqslant -1} P_n^\bullet$. Then*

$$M^\bullet \otimes^{\mathbf{L}} N^\bullet \simeq \operatorname*{colim}_{n,m \geqslant -1} (P_n^\bullet \otimes \tau^{\leqslant m} N^\bullet) \simeq P^\bullet \otimes N^\bullet. \tag{2.57}$$

2.2 Monads and descent

I assume that the reader is familiar with the notion of a monad in ordinary category theory, and a *module* over it[4]. In short, given a an ordinary category $\mathscr{C}$, the category $\mathrm{End}(\mathscr{C})$ of endofunctors is naturally a monoidal category via composition of functors. Monoids in $\mathrm{End}(\mathscr{C})$ are known as *monads.* To give such an object is the same as giving an endofunctor T together with *multiplication* and *unit* transformations $\mu : T^2 \to T$ and $\eta : \mathrm{id} \to T$ satisfying diagrams expressing strict associativity and unitality. Evaluation on objects gives an action $\mathrm{End}(\mathscr{C}) \times \mathscr{C} \to \mathscr{C}$ of this monoidal category on $\mathscr{C}$. The notion of a module over an algebra object makes sense in this generality.[5] In particular, given

[4]In this paper we prefer to use the terminology of *modules over monads* rather than *algebras over monads.*

[5]That is, whenever one has a monoidal category $\mathscr{D}$ acting on a category $\mathscr{C}$, one can consider, for any algebra object $T \in \mathscr{D}$, the category $\mathrm{Mod}_T \mathscr{C}$ of module objects in $\mathscr{C}$ over T. In particular, it is not necessary for the "module" and the "algebra" to belong to the same category.

any monad $T \in \mathscr{C}$, we obtain the category $\mathrm{Mod}_T \mathscr{C}$ of modules over the monad. To give such an object is the same as giving an object $M \in \mathscr{C}$ together with an action morphism $TM \to M$ satisfying a diagram expressing the module axiom. The definition of a comonad, and a comodule over a comonad, is obtained by formally dualizing these definitions.

An abundant source of (co)monads in ordinary category theory, is from adjunctions. Given an adjunction $F : \mathscr{D} \leftrightarrows \mathscr{C} : G$, the endofunctor FG on $\mathscr{C}$ naturally acquires the structure of a monad with unit $\eta : \mathrm{id} \to FG$ acquired from the unit transformation and multiplication $\mu : FGFG \to FG$ acquired from the counit transformation $\varepsilon : GF \to \mathrm{id}$. The functor G canonically factors through the forgetful functor as $\mathscr{C} \xrightarrow{K} \mathrm{Mod}_{FG} \mathscr{D} \to \mathscr{D}$. The functor K is called the *comparison functor*, and the classical theorem of Barr-Beck gives neccesary and sufficient conditions for K to be an equivalence, in which case we say $F \dashv G$ is *monadic*. That is, G should be conservative and preserve certain kinds of colimits, called G-split coequalizers.[6]

The comonadicity of adjunctions is related to descent theory via the classical Bénabou-Roubaud theorem, which roughly says that, under a hypothesis related to base-change, (which is sometimes called the Beck-Chevalley condition), descent is “the same” as the comonadicity of the push-pull adjunction.

In this subsection, I will attempt, in analogy with this story, to describe how modules over a *homotopy-coherent monad* are constructed in higher category theory. I will also record some results about the relation between monads and descent theory.

2.2.1 What are modules over a homotopy-coherent monad?

I am including this section for completeness, but I cannot pretend to give a better introduction than the notes of Lukas Brantner [Bra24], which were enormously helpful for me when learning this. We recall that ∞-categories are built out of the category $s\mathsf{Set}$ simplicial sets. More precisely, they are the fibrant objects in the Joyal model structure on $s\mathsf{Set}$ [Lur09b, §2.2.5]. By using the Cartesian-closed structure[7] on $s\mathsf{Set}$, one can, to each ∞-category $\mathscr{C}$, define another ∞-category $\mathrm{End}(\mathscr{C}) := \mathscr{C}^{\mathscr{C}}$. By again using the Cartesian-closed structure on $s\mathsf{Set}$ one obtains a diagram $N(\Delta^{\mathsf{op}}) \to s\mathsf{Set}$ sending $[n] \mapsto \mathrm{End}(\mathscr{C})^n$,[8], c.f. [Bra24, Definition 2.46]. By the straightening/unstraightening equivalence[9] this can be equivalently viewed as a coCartesian fibration[10] $\mathrm{End}(\mathscr{C})^{\otimes} \to N(\Delta^{\mathsf{op}})$. Here $\mathrm{End}(\mathscr{C})^{\otimes}$ is defined by unstraightening applied to the previous construction. This fibration turns out to be a *monoidal ∞-category*, i.e., it is an inner fibration[11] of simplicial sets satisfying the Segal property.[12] When referring to this monoidal ∞-category we may suppress the implicit fibration or even also the $\otimes$.

Definition 2.2.1. *[Lur09a, Definition 1.1.14, Definition 3.1.3] A* homotopy-coherent monad*, which we will often just call a* monad*, is defined to be an algebra object in*

[6] Of course, the Barr-Beck theorem also has a formal dual for comonadic adjunctions.

[7] That is, given $X, Y \in s\mathsf{Set}$ one may define $Y^X \in s\mathsf{Set}$ by $(Y^X)_n := \mathrm{Map}_{s\mathsf{Set}}(\Delta^n \times X, Y)$, and there is an adjunction $(-) \times X \dashv (-)^X$.

[8] So that $[0] \mapsto [0] \in s\mathsf{Set}$.

[9] *Straightening/unstraightening* is the common name given to the *Grothendieck construction* for $(\infty, 1)$ categories [Lur09b, §3.2]. It asserts that there is a Quillen equivalence between the category of simplicial functors, and coCartesian fibrations.

[10] A coCartesian fibration is the higher categorical counterpart of a Grothendieck op-fibration, see [Lur09b, Definition 2.4.2.1].

[11] A morphism of simplicial sets is called an *inner fibration* if it has the left lifting property against inclusions of *inner* horns.

[12] For details we direct the reader to [Lur17, Ch. 2] and [Lur09a, Ch. 1].

the monoidal ∞-category $\mathrm{End}(\mathscr{C})^{\otimes}$. *This means that it is a section of the fibration* $\mathrm{End}(\mathscr{C})^{\otimes} \to N(\Delta^{\mathsf{op}})$ *sending the* inert morphisms[13] *in* $N(\Delta^{\mathsf{op}})$ *to coCartesian edges.*

Informally, a homotopy-coherent monad is an endofunctor $T \in \mathrm{End}(\mathscr{C})$ equipped with morphisms $\mu : T^2 \to T$ and $\eta : \mathrm{id} \to T$ which are associative and unital up to coherent homotopy.

In higher-category theory, the action of the monoidal ∞-category $\mathrm{End}(\mathscr{C})^{\otimes}$ on $\mathscr{C}$ is captured via the definition of a *left-tensored ∞-category* [Lur09a, Definition 2.1.1]. In this situation this means the following, c.f. [Lur09a, Proposition 3.1.2]. We will construct an ∞-category $\mathscr{C}^{\otimes}$ equipped with a coCartesian fibration $\mathscr{C}^{\otimes} \to N(\Delta^{\mathsf{op}})$ and a fibration[14] $\mathscr{C}^{\otimes} \to \mathrm{End}(\mathscr{C})^{\otimes}$ over $N(\Delta^{\mathsf{op}})$ such that, for every $n \geqslant 0$ the inclusion $\{n\} \to [n]$ induces an equivalence $\mathscr{C}^{\otimes}_{[n]} \xrightarrow{\sim} \mathrm{End}(\mathscr{C})^{\otimes}_{[n]} \times \mathscr{C}$; here $(-)_{[n]}$ denotes the fiber over $[n] \in \Delta^{\mathsf{op}}$. In order to define $\mathscr{C}^{\otimes}$ we note that the Cartesian-closed structure on $s\mathsf{Set}$ gives a diagram $N(\Delta^{\mathsf{op}}) \to s\mathsf{Set}$ sending $[n] \mapsto \mathrm{End}(\mathscr{C})^n \times \mathscr{C}$. By using the morphism $\mathscr{C} \to [0]$ to the terminal object and unstraightening we obtain ∞-category $\mathscr{C}^{\otimes}$ equipped with the desired fibration $\mathscr{C}^{\otimes} \to \mathrm{End}(\mathscr{C})^{\otimes}$ over $N(\Delta^{\mathsf{op}})$.

Definition 2.2.2. *[Lur09a, Definition 2.1.4, Definition 3.1.3] Fix a homotopy-coherent monad T, viewed as a section $T : N(\Delta^{\mathsf{op}}) \to \mathrm{End}(\mathscr{C})^{\otimes}$. A* module over T *is a section* $M : N(\Delta^{\mathsf{op}}) \to \mathscr{C}^{\otimes}$ *such that:*

- ⋆ *The composite* $N(\Delta^{\mathsf{op}}) \to \mathscr{C}^{\otimes} \to \mathrm{End}(\mathscr{C})^{\otimes}$ *is equivalent to* T,
- ⋆ M *sends edges corresponding to* convex morphisms[15] $\alpha : [m] \to [n]$ *in* Δ *such that* $\alpha(m) = n$, *to coCartesian edges for the fibration* $\mathscr{C}^{\otimes} \to N(\Delta^{\mathsf{op}})$.

Informally [Lur09a, Remark 3.1.4] a module M over a homotopy-coherent monad is an object $M \in \mathscr{C}$ equipped with a morphism $TM \to M$ which satisfies a version of the module axiom up to coherent homotopy.

We would like to produce homotopy-coherent monads from adjunctions. The usual definition of an *adjunction* in higher category theory uses not much data. That is, an adjunction consists of a pair of functors $F : \mathscr{D} \leftrightarrows \mathscr{C} : G$ of ∞-categories together with a unit $\eta : \mathrm{id} \to FG$, a counit $\varepsilon : FG \to \mathrm{id}$ and a 2-simplex expressing the zig-zag identity for the composite $F \to FGF \to F$, c.f. [Lur18a, Tag 02EJ]. *A priori*, it is not clear how to produce enough coherences in order to give the endofunctor FG the structure of a homotopy-coherent monad. This motivates the definition of the ∞-category of *adjunction data* [Lur09a, §3.2]. The ∞-category $\mathrm{ADat}(\mathscr{D},\mathscr{C})$ consists of certain sections of a certain fibration over a version of Δ^{op} labelled by two colours; we direct the reader to [Lur09a, Definition 3.2.6] and [Bra24, Lecture 3] for details. This category keeps track of all the possible coherences implicit in an adjunction, so that every object of $\mathrm{ADat}(\mathscr{D},\mathscr{C})$ gives rise to a homotopy-coherent monad on $\mathscr{C}$, c.f. [Lur09a, Remark 3.2.7]. There is a functor $\mathrm{ADat}(\mathscr{D},\mathscr{C}) \to \mathrm{Fun}^L(\mathscr{D},\mathscr{C})$ from the ∞-category of adjunction data to the category of left-adjoint functors, in the previous sense. By Lurie's Theorem on the existence of adjunction data [Lur09a, Theorem 3.2.10], this is a *trivial Kan fibration.* That is, given any left-adjoint functor $F : \mathscr{D} \to \mathscr{C}$ one can, (up to contractible choice), choose a right adjoint G such that the endofunctor FG acquires the structure of a homotopy-coherent monad. Such a functor G can then be factored canonically through the forgetful functor as $\mathscr{C} \xrightarrow{K} \mathrm{Mod}_{FG}\,\mathscr{D} \to \mathscr{D}$, c.f. [Lur17, §4.7]. Let us call the functor K the *comparison functor.*

[13] See [Bra24, Lecture 2, Definition 2.48].
[14] In the Joyal model structure.
[15] See [Lur09a, Definition 1.1.7], and also [Bra24, Lecture 2, Definition 2.54].

Theorem 2.2.3 (Barr–Beck–Lurie)**.** *[Lur17, §4.7] With notations as above. Assume that G is conservative and that G* preserves geometric realizations of G-split simplicial objects[16]*. Then the comparison functor K is an equivalence.*

The equivalence of categories in the Barr–Beck–Lurie theorem can be made more explicit in the following way:

Lemma 2.2.4. *(i) Let $F : \mathscr{C} \leftrightarrows \mathscr{D} : G$ be a comonadic adjunction of ∞-categories. The quasi-inverse to the comparison equivalence $\mathscr{C} \xrightarrow{\sim} \mathrm{Comod}_{FG}\,\mathscr{D}$, is given on objects by the formula*

$$M \mapsto \lim_{[n]\in\Delta} G(FG)^n M. \tag{2.58}$$

(ii) Let $G : \mathscr{D} \leftrightarrows \mathscr{C} : F$ be a monadic adjunction of ∞-categories. The quasi-inverse to the comparison equivalence $\mathscr{C} \simeq \mathrm{Mod}_{FG}\,\mathscr{D}$, is given on objects by the formula

$$M \mapsto \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} G(FG)^n M. \tag{2.59}$$

Proof. We will prove (ii), the proof of (i) is similar. We examine the equivalence of categories implicit in the proof of the Barr-Beck-Lurie theorem. Let us temporarily write T for the monad FG. The Barr-Beck-Lurie equivalence arises from an adjunction

$$R : \mathscr{C} \leftrightarrows \mathrm{Mod}_T(\mathscr{D}) : L \tag{2.60}$$

in which R satisfies

$$\mathrm{Forget}_T \circ R \simeq F. \tag{2.61}$$

We would like to calculate the value of L on objects. Every $M \in \mathrm{Mod}_T(\mathscr{D})$ is the colimit of simplicial object

$$M \simeq \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} T^{n+1} M \tag{2.62}$$

in $\mathrm{Mod}_T(\mathscr{D})$. Since L is a left adjoint, it commutes with colimits, so it suffices to know the value of L on free T-modules. However, by passing to left adjoints in (2.61), we know that

$$L \circ \mathrm{Free}_T \simeq G, \tag{2.63}$$

so $LM \simeq \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} GT^n M = \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} G(FG)^n M$. This proves (ii). □

2.2.2 Barr–Beck–Lurie in families

In this section we present a generalization of the result of [GHK22, Proposition 4.4.5] which is adapted to our setting.

Proposition 2.2.5. *Given a diagram*

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{U} & \mathscr{D} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle r} & \\ & \mathscr{B} & \end{array} \tag{2.64}$$

in Cat_∞ such that:

(i) p and r are coCartesian fibrations and U preserves coCartesian edges;

[16]See [Lur17, §4.7].

(ii) U has a left adjoint $F : \mathscr{D} \to \mathscr{C}$ such that $pF \simeq r$;

(iii) The adjunction $F \dashv U$ restricts in each fiber to an adjunction $F_b \dashv U_b$. For all $b \in \mathscr{B}$, the functor U_b is conservative, and $\mathscr{C}_b$ admits colimits of U_b-split simplicial objects, which U_b preserves.

(iv) For any edge $e : b \to b'$ in $\mathscr{B}$, the coCartesian covariant transport $e_! : \mathscr{C}_b \to \mathscr{C}_{b'}$ preserves geometric realizations of U_b-split simplicial objects.

Then, the adjunction $F \dashv U$ is monadic.

Remark 2.2.6. *In view of the Barr–Beck–Lurie theorem, condition (iii) in Proposition 2.2.5 is equivalent to:*

(iii)′ The adjunction $F \dashv U$ restricts in each fiber to a monadic adjunction $F_b \dashv U_b$.

Proof of Proposition 2.2.5. We verify the conditions of the Barr–Beck–Lurie theorem [Lur17, Theorem 4.7.3.5].

First we show that U is conservative. We can argue in exactly the same way as [GHK22, Proposition 4.4.5]. Suppose that $f : c \to c'$ is a morphism in $\mathscr{C}$ such that Uf is an equivalence in $\mathscr{D}$. Then $e := qUf \simeq pf$ is an equivalence in $\mathscr{B}$. One can factor f as $c \xrightarrow{\varphi} e_!c \xrightarrow{f'} c'$ where φ is a coCartesian lift of e and f' is a morphism in the fiber $\mathscr{C}_{b'}$ above $b' := p(c')$. Since φ is coCartesian lift of an equivalence, it is an equivalence. Because of the fiberwise monadicity assumption (iii), f' is an equivalence. Therefore f is an equivalence and U is conservative.

Now let us show that $\mathscr{C}$ admits and U preserves colimits of U-split simplicial objects. Let $q : \Delta^{\mathsf{op}} \to \mathscr{C}$ be a U-split simplicial object, so that Uq extends to a diagram $\widetilde{Uq} : \Delta^{\mathsf{op}}_{-\infty} \to \mathscr{D}$. Let $f : \Delta^{\mathsf{op}}_{-\infty} \to \mathscr{B}$ be the underlying diagram in $\mathscr{B}$. There is a morphism

$$\Delta^1 \times \Delta^{\mathsf{op}}_{-\infty} \to \Delta^{\mathsf{op}}_{-\infty} \tag{2.65}$$

which is the identity on $\{0\} \times \Delta^{\mathsf{op}}_{-\infty}$ and carries $\{1\} \times \Delta^{\mathsf{op}}_{-\infty}$ to $[-1] \in \Delta^{\mathsf{op}}_{-\infty}$. It sends each horizontal morphism $\{0\} \times [n] \to \{1\} \times [n]$ to the unique morphism $[n] \to [-1]$. Consider the composite

$$P : \Delta^1 \times \Delta^{\mathsf{op}}_{-\infty} \to \Delta^{\mathsf{op}}_{-\infty} \xrightarrow{f} \mathscr{B}. \tag{2.66}$$

Now we will take a coCartesian lifts, using the exponentiation for coCartesian fibrations [Lur18a, Tag 01VG].

- ⋆ Let Q be a coCartesian lift of $P|_{\Delta^1 \times \Delta^{\mathsf{op}}}$ to $\mathscr{C}$. Then Q is a natural transformation between q and a morphism $q' : \Delta^{\mathsf{op}} \to \mathscr{C}_b$, where b is the image under f of $[-1] \in \Delta^{\mathsf{op}}_{-\infty}$.
- ⋆ Let $\widetilde{UQ}$ be a coCartesian lift of P to $\mathscr{D}$. Then $\widetilde{UQ}$ is a natural transformation between $\widetilde{Uq}$ and a morphism $\widetilde{Uq'} : \Delta^{\mathsf{op}}_{-\infty} \to \mathscr{C}_b$.

These natural transformations Q and $\widetilde{UQ}$ are uniquely characterised by the property that their components are coCartesian edges [Lur18a, Tag 01VG]. Because of the assumption (i) that U preserves coCartesian edges, this unicity implies that $UQ \simeq \widetilde{UQ}\Big|_{\Delta^1 \times \Delta^{\mathsf{op}}}$. In particular $Uq' : \Delta^{\mathsf{op}} \to \mathscr{C}_b$ extends to the split simplicial object $\widetilde{Uq'} : \Delta^{\mathsf{op}}_{-\infty} \to \mathscr{C}_b$. By the fiberwise monadicity assumption (iii), this implies that q' extends to a colimit diagram $\overline{q}' : (\Delta^{\mathsf{op}})^{\triangleright} \to \mathscr{C}_b$ such that $U\overline{q}'$ is also a colimit diagram. By assumption (iv) and [Lur09b, Proposition 4.3.1.10] it then follows that $\overline{q}'$ (resp. $U\overline{q}'$), when regarded as a diagram in $\mathscr{C}$ (resp. $\mathscr{D}$), is a p-colimit diagram (resp. r-colimit diagram). Now we can argue as in [Lur09b, Corollary 4.3.1.11]. We have a commutative diagram

$$\begin{array}{ccc} (\Delta^1 \times \Delta^{\mathrm{op}}) \coprod_{\{1\}\times\Delta^{\mathrm{op}}} (\{1\} \times (\Delta^{\mathrm{op}})^{\triangleright}) & \xrightarrow{(Q,\overline{q}')} & \mathscr{C} \\ \big\downarrow & \overset{s}{\nearrow} & \big\downarrow p \\ (\Delta^1 \times \Delta^{\mathrm{op}})^{\triangleright} & \xrightarrow[(f|_{(\Delta^{\mathrm{op}})^{\triangleright}})\circ\pi]{} & \mathscr{B} \end{array} \tag{2.67}$$

in which $\pi : (\Delta^1 \times \Delta^{\mathrm{op}})^{\triangleright} \to (\Delta^{\mathrm{op}})^{\triangleright} = \Delta^{\mathrm{op}}_{+} \subseteq \Delta^{\mathrm{op}}_{-\infty}$ denotes the morphism which is the identity on $\{0\} \times \Delta^{\mathrm{op}}$ and which carries $(\{1\} \times \Delta^{\mathrm{op}})^{\triangleright}$ to the cone point. Because the left map is an inner fibration there exists a lift s as indicated by the dashed arrow. Consider now the map $\Delta^1 \times (\Delta^{\mathrm{op}})^{\triangleright} \to (\Delta^1 \times \Delta^{\mathrm{op}})^{\triangleright}$ which is the identity on $\Delta^1 \times \Delta^{\mathrm{op}}$ and carries the other vertices of $\Delta^1 \times (\Delta^{\mathrm{op}})^{\triangleright}$ to the cone point. Let $\overline{Q}$ denote the composition

$$\Delta^1 \times (\Delta^{\mathrm{op}})^{\triangleright} \to (\Delta^1 \times \Delta^{\mathrm{op}})^{\triangleright} \xrightarrow{s} \mathscr{C} \tag{2.68}$$

and define $\overline{q} := \overline{Q}|_{\{0\}\times(\Delta^{\mathrm{op}})^{\triangleright}}$. Then $\overline{Q}$ is a natural transformation from $\overline{q}$ to $\overline{q}'$ which is componentwise coCartesian. Then [Lur09b, Proposition 4.3.1.9] implies that $\overline{q}$ is a p-colimit diagram which fits into the diagram

$$\begin{array}{ccc} \Delta^{\mathrm{op}} & \xrightarrow{q} & \mathscr{C} \\ \big\downarrow & \overset{\overline{q}}{\nearrow} & \big\downarrow p \\ (\Delta^{\mathrm{op}})^{\triangleright} & \xrightarrow[f|_{(\Delta^{\mathrm{op}})^{\triangleright}}]{} & \mathscr{B} \end{array} \tag{2.69}$$

By assumption (i), $U\overline{Q}$ is a natural transformation from $U\overline{q}$ to $U\overline{q}'$ which is componentwise coCartesian. Hence [Lur09b, Proposition 4.3.1.9] implies that $U\overline{q}$ is an r-colimit diagram. The underlying diagram $f|_{(\Delta^{\mathrm{op}})^{\triangleright}}$ of $\overline{q}$ in $\mathscr{B}$ extends to the split simplicial diagram f and hence admits a colimit in $\mathscr{B}$. Hence [Lur09b, Proposition 4.3.1.5(2)] implies that $\overline{q}$ and $U\overline{q}$ are colimit diagrams in $\mathscr{C}$ and $\mathscr{D}$ respectively. Hence $\mathscr{C}$ admits and U preserves geometric realizations of U-split simplicial objects. ☐

Corollary 2.2.7. *Let $\mathscr{B}$ be an ∞-category and let $\eta : F \to G$ be a natural transformation of functors $F, G : \mathscr{B} \to \mathsf{Cat}_\infty$. Assume that:*

- *(i) For each $b \in \mathscr{B}$ the functor $\eta_b : F(b) \to G(b)$ admits a left adjoint ν_b;*
- *(ii) For each $b \in \mathscr{B}$, the functor η_b is conservative, and $F(b)$ admits colimits of η_b-split simplicial objects, which η_b preserves;*
- *(iii) For any edge $e : b \to b'$ in $\mathscr{B}$, the functor $F(e) : F(b) \to F(b')$ preserves geometric realizations of η_b-split simplicial objects.*

Then there exists a functor $H : \mathscr{B} \to \mathsf{Cat}_\infty$ equipped with a natural equivalence $\lambda : F \xrightarrow{\sim} H$ and a natural transformation $\mu : H \to G$ such that:

- *(i) There is an equivalence $\mu \circ \lambda \simeq \eta$;*

(ii) Set $T_b := \eta_b\nu_b$. Then for each $b \in \mathscr{B}$ one has $H(b) = \mathrm{Mod}_{T_b} F(b)$ and

$$F(b) \xrightarrow{\simeq\lambda(b)} H(b) \xrightarrow{\mu(b)} G(b) \tag{2.70}$$

identifies with the factorization of $\eta(b)$ as the comparison functor *followed by the* forgetful functor.

Proof. This follows from Proposition 2.2.5 and straightening. Strictly speaking, an application of (the dual of) [Lur17, Proposition 7.3.2.6] is required. ☐

2.2.3 Noncommutative notion of descendability

In this section we present a mild generalization of the results of [Mat16]. Let $\mathscr{V}$ be a a presentably symmetric monoidal stable ∞-category. Let $\mathrm{Alg}(\mathscr{V})$ be the category of associative algebra objects in $\mathscr{V}$. The constructions in this section are phrased in terms of *left* modules, but have obvious counterparts for right modules. As usual, for an algebra object $A \in \mathsf{Alg}(\mathscr{V})$, we denote by $\mathrm{LMod}_A := \mathrm{LMod}_A \mathscr{V}$ the ∞-category of *left module objects* over A [Lur17, §4.2]. Let $f^\# : A \to B$ be a morphism in $\mathrm{Alg}(\mathscr{V})$. We obtain a forgetful functor

$$f_* : \mathrm{LMod}_B \to \mathrm{LMod}_A\,. \tag{2.71}$$

By [Lur17, Corollary 4.2.3.2], there is a functor $\mathrm{Alg}(\mathscr{V})^{\mathsf{op}} \to \mathsf{Cat}_\infty$ which sends $A \mapsto \mathrm{LMod}_A$ and each $f^\# : A \to B$ to the forgetful functor f_*.

The functor f_* has a left adjoint by the following construction. We may regard B as an object of ${}_B\mathrm{BMod}_B$, the category of B-B bimodule objects in $\mathscr{V}$. (In this section, we will always regard bimodule objects as a monoidal category with respect to *convolution* [Lur17, §4.4.3]). There is a functor

$${}_B\mathrm{BMod}_B \to_B \mathrm{BMod}_A \simeq \mathrm{Fun}^L_{\mathscr{V}}(\mathrm{LMod}_A, \mathrm{LMod}_B), \tag{2.72}$$

the image of B under this composite is the left adjoint f^* to f_*. By [Lur17, Proposition 4.6.2.17] the formation of these pullback functors can be arranged in a functorial way. Accordingly we obtain a comonad $f^*f_* = B \otimes_A -$ on LMod_B.

Definition 2.2.8. *With notations as above. We let $\langle B\rangle$ denote smallest full subcategory of ${}_A\mathrm{BMod}_A$ which contains B and is stable under finite (co)limits, retracts, and convolution. We say that $f^\# : A \to B$ is* descendable *if $A \in \langle B\rangle$.*

Lemma 2.2.9. *Suppose that $f^\# : A \to B$ is descendable. Then the adjunction $f^* \dashv f_*$ is comonadic.*

Before the proof of this Lemma we recall some properties of Pro- and Ind-objects.

Definition 2.2.10. *[Mat16, §3] Let $\mathscr{C}$ be an ∞-category with finite limits and colimits. Let $\mathrm{Ind}(\mathscr{C})$ and $\mathrm{Pro}(\mathscr{C}) := \mathrm{Ind}(\mathscr{C}^{\mathsf{op}})^{\mathsf{op}}$ be the* Pro- *and* Ind-*categories of $\mathscr{C}$, respectively.*

(i) An object $M \in \mathrm{Ind}(\mathscr{C})$ is called a constant Ind-object *of $\mathscr{C}$ if it belongs to the essential image of the Yoneda embedding $\mathscr{C} \to \mathrm{Ind}(\mathscr{C})$.*

(ii) An object $M \in \mathrm{Pro}(\mathscr{C})$ is called a constant Pro-object *of $\mathscr{C}$ if it belongs to the essential image of the Yoneda embedding $\mathscr{C} \to \mathrm{Pro}(\mathscr{C})$.*

Lemma 2.2.11. *[Mat16, §3]. Let $\mathscr{C}$ be an ∞-category with finite limits and colimits.*

(i) Let $p : I \to \mathscr{C}$ be a filtered diagram. The following are equivalent:

(a) The object "colim"$p \in \mathrm{Ind}(\mathscr{C})$ *is a constant* Ind-*object;*

(b) For every ∞-category $\mathscr{D}$ and every functor $F : \mathscr{C} \to \mathscr{D}$ which preserves finite colimits, the colimit of p is preserved by F.

(ii) Let $p : I \to \mathscr{C}$ be a cofiltered diagram. The following are equivalent:

(a) The object "lim"$p \in \mathrm{Pro}(\mathscr{C})$ *is a constant* Pro-*object;*

(b) For every ∞-category $\mathscr{D}$ and every functor $F : \mathscr{C} \to \mathscr{D}$ which preserves finite limits, the limit of p is preserved by F.

Example 2.2.12. *[Mat16, Example 3.11]. Let $\mathscr{C}$ be an ∞-category with finite limits and colimits.*

(i) If $M_\bullet$ is a split simplicial object of $\mathscr{C}$, then the system of its n-skeleta $\mathrm{sk}_n(M_\bullet) := \mathrm{colim}_{[m]\in\Delta^{\mathrm{op}}_{\leqslant n}} M_m$ *forms a constant* Ind-*object with* $\mathrm{colim}_n \mathrm{sk}_n(M_\bullet) = \mathrm{colim}_{[m]\in\Delta^{\mathrm{op}}} M_\bullet$.

(ii) If $M^\bullet$ is a split cosimplicial object of $\mathscr{C}$, then the system of its partial totalizations $\mathrm{Tot}_n(M^\bullet) := \lim_{[m]\in\Delta_{\leqslant n}} M^m$ *forms a constant* Pro-*object with* $\lim_n \mathrm{Tot}_n(M^\bullet) = \lim_{[m]\in\Delta} M^\bullet$.

Proof of Lemma 2.2.9. This is essentially the same as [Man22, Proposition 2.6.3]. We check the hypotheses of the Barr–Beck–Lurie theorem. Let $M^\bullet \in \mathrm{LMod}_A$ be a f^*-split cosimplicial object. In particular the Tot-tower of $f_*f^*M^\bullet = B \otimes_A M^\bullet$ is a constant Pro-object. It follows (using that $\mathscr{V}$ is stable) that the Tot-tower of $N \otimes_A M^\bullet$ is constant for every $N \in \langle B\rangle$. By taking $N = A \in \langle B\rangle$ one deduces that the Tot-tower of $M^\bullet$ is a constant Pro-object. In this case it is clear that f^* commutes with the totalization of $M^\bullet$: as the relevant categories are stable, f^* preserves finite limits.

It remains to check that f^* is conservative. Suppose that $f^*M \simeq 0$, then $f_*f^*M = B \otimes_A M \simeq 0$ and hence $N \otimes_A M \simeq 0$ for every $N \in \langle B\rangle$. Taking $N = A \in \langle B\rangle$ then implies that $M \simeq 0$. ☐

Now let $A^\bullet$ be an augmented cosimplicial object in $\mathrm{Alg}(\mathscr{V})$. We obtain a functor

$$N(\Delta_+) \to \mathsf{Cat}_\infty : [n] \mapsto \mathrm{LMod}_{A^n}, \tag{2.73}$$

which sends every morphism $[m] \to [n]$ in Δ_+ to the corresponding pullback functor $\mathrm{LMod}_{A^m} \to \mathrm{LMod}_{A^n}$ defined as in the preceding section. In the next definition we will use the following notation. For each $[n] \in \Delta_+$ we denote by d^0 the injective morphism $d^0 : [n] \hookrightarrow [0]\star[n] \simeq [n+1]$ which omits 0 from its image. For any morphism $\alpha : [m] \to [n]$ in Δ_+ there is an obvious cosimplicial morphism $\alpha' : [m+1] \to [n+1]$ such that $\alpha' d^0 = d^0\alpha$.

Definition 2.2.13. *We say that $A^\bullet$* satisfies the Beck-Chevalley condition *if for every map $\alpha : [m] \to [n]$ in Δ_+, the natural morphism*

$$A^n \otimes_{A^m} A^{m+1} \to A^{n+1} \tag{2.74}$$

is an equivalence in ${}_{A^n}\mathrm{BMod}_{A^{m+1}}$. Here, the algebra morphisms $A^m \to A^{m+1}$ and $A^n \to A^{n+1}$ are induced by d^0. The morphisms $A^m \to A^n$ and $A^{m+1} \to A^{n+1}$ are induced by α and α'.

Lemma 2.2.14. *Suppose that $A^\bullet$ satisfies the Beck-Chevalley condition and $A^{-1} \to A^0$ is descendable. Then the canonical morphism*

$$\mathrm{LMod}_{A^{-1}} \to \lim_{[n]\in\Delta} \mathrm{LMod}_{A^n} \tag{2.75}$$

is an equivalence of ∞-categories.

Proof. Follows from Lemma 2.2.9 above and [Lur17, Corollary 4.7.5.3]. □

Example 2.2.15. *Suppose that $A \to B$ is a morphism of* commutative *algebra objects in $\mathscr{V}$. Then $A^{\otimes_B(\bullet+1)}$ is an augmented cosimplicial algebra object which satisfies the Beck–Chevalley condition. It is obvious that $A \to B$ is descendable in the sense of Definition 2.2.8 if and only if $A \to B$ is descendable in the sense of Mathew [Mat16, §3.3]. Hence Lemma 2.2.14 recovers Mathew's theorem [Mat16, Proposition 3.22].*

2.2.4 A way to produce monoids from monads

For an operad $\mathscr{O}$ we let $\mathsf{CatMon}^{\mathrm{lax}}_{\mathscr{O}}$ denote the $(\infty,2)$-category of $\mathscr{O}$-monoidal categories with lax $\mathscr{O}$-linear functors [HHLN23, Definition 3.4.1] and $\mathsf{CatMon}^{\mathrm{oplax}}_{\mathscr{O}}$ denote the $(\infty,2)$-category of $\mathscr{O}$-monoidal categories with oplax $\mathscr{O}$-linear functors [HHLN23, Definition 3.4.3].

Let Assoc denote the associative operad [Lur17, Definition 4.1.1.3] and let LM denote the operad of [Lur17, Notation 4.2.1.6]. Let $\mathscr{V}$ be a monoidal $(\infty,1)$-category which we identify with the object $\mathscr{V} \to \mathsf{Assoc}$ of $\mathsf{CatMon}^{\mathrm{lax}}_{\mathsf{Assoc}}$. By abuse of notation we also identify $\mathscr{V}$ with the object $\mathscr{V}^{\mathsf{op}} \to \mathsf{Assoc}^{\mathsf{op}}$ of $\mathsf{CatMon}^{\mathrm{oplax}}_{\mathsf{Assoc}}$.

Definition 2.2.16. *(i) The $(\infty,2)$-category of $\mathscr{V}$-linear categories with* lax *$\mathscr{V}$-linear functors is defined to be*

$$\mathrm{LMod}^{\mathrm{lax}}_{\mathscr{V}} := \{\mathscr{V}\} \times_{\mathsf{CatMon}^{\mathrm{lax}}_{\mathsf{Assoc}}} \mathsf{CatMon}^{\mathrm{lax}}_{\mathsf{LM}}. \tag{2.76}$$

(ii) The $(\infty,2)$-category of $\mathscr{V}$-linear categories with oplax *$\mathscr{V}$-linear functors is defined to be*

$$\mathrm{LMod}^{\mathrm{oplax}}_{\mathscr{V}} := \{\mathscr{V}\} \times_{\mathsf{CatMon}^{\mathrm{oplax}}_{\mathsf{Assoc}}} \mathsf{CatMon}^{\mathrm{oplax}}_{\mathsf{LM}}. \tag{2.77}$$

Here is an intuitive description. A *$\mathscr{V}$-linear ∞-category* is an ∞-category $\mathscr{C}$ equipped with the data of a functor $\otimes : \mathscr{V} \times \mathscr{C} \to \mathscr{C}$ which is unital and associative up to coherent homotopy. A *lax $\mathscr{V}$-linear functor* $\mathscr{C} \to \mathscr{D}$ of $\mathscr{V}$-linear ∞-categories $\mathscr{C}, \mathscr{D}$ is a functor $F : \mathscr{C} \to \mathscr{D}$ equipped with the data of morphisms $V \otimes F(M) \to F(V \otimes M)$, for all $V \in \mathscr{V}, M \in \mathscr{C}$, which is unital and associative up to coherent homotopy. Oplax linear functors are defined similarly but with arrows reversed.

Theorem 2.2.17. *[HHLN23, Theorem C] Let $\mathrm{LMod}^{R,\mathrm{lax}}_{\mathscr{V}}$ denote the 1-full 2-subcategory of $\mathrm{LMod}^{\mathrm{lax}}_{\mathscr{V}}$ spanned by those functors which are objectwise right adjoints. Let $\mathrm{LMod}^{L,\mathrm{oplax}}_{\mathscr{V}}$ denote the 1-full subcategory of $\mathrm{LMod}^{\mathrm{oplax}}_{\mathscr{V}}$ spanned by those functors which are objectwise left adjoints. Then there is an equivalence of $(\infty,2)$-categories*

$$\mathrm{LMod}^{R,\mathrm{lax}}_{\mathscr{V}} \simeq \left(\mathrm{LMod}^{L,\mathrm{oplax}}_{\mathscr{V}}\right)^{(1,2)-\mathsf{op}}, \tag{2.78}$$

obtained by passing to adjoints objectwise (in both directions). Here $(1,2)-\mathsf{op}$ indicates that the direction of 1- and 2-morphisms are reversed.

Proof. This follows immediately from [HHLN23, Theorem C], taking the operad $\mathscr{O}$ in *loc. cit.* to be LM and Assoc and then taking the fiber over $\mathscr{V}$[17]. □

Intuitively, this may be explained as follows. Let $\mathscr{C}$ and $\mathscr{D}$ be $\mathscr{V}$-linear categories. Let $F : \mathscr{D} \leftrightarrows \mathscr{C} : G$ be an adjunction in which the left adjoint F is oplax linear. Let $V \in \mathscr{V}, M \in \mathscr{C}$. The desired morphism $V \otimes G(M) \to G(V \otimes M)$ giving the lax linear structure on G is adjoint to the composite

$$F(V \otimes G(M)) \to V \otimes FG(M) \to V \otimes M, \tag{2.79}$$

where the first is from the oplax linear structure on F and the second is induced by the counit of $F \dashv G$. Similarly, a lax linear structure on G determines an oplax linear structure on F. The Theorem says that these operations give a mutually inverse equivalence of $(\infty, 2)$-categories.

Lemma 2.2.18. *Let us view $\mathscr{V}$ as a $\mathscr{V}$-linear category. Then there is an adjunction of ∞-categories*

$$\iota : \mathscr{V} \leftrightarrows \mathrm{Fun}^{\mathrm{lax}}_{\mathscr{V}}(\mathscr{V}, \mathscr{V}) : \kappa \tag{2.80}$$

in which the left adjoint ι sends $V \mapsto - \otimes V$ and the right adjoint κ sends $F \mapsto F(1_{\mathscr{V}})$. (Note here that the superscript lax *stands for* lax $\mathscr{V}$-linear *functors and* not *lax monoidal functors).*

Proof. The unit morphism $\eta : \mathrm{id} \xrightarrow{\sim} \kappa\iota$ is the canonical equivalence $\mathrm{id} \simeq 1_{\mathscr{V}} \otimes \mathrm{id}$. The counit morphism $\varepsilon : \iota\kappa \to \mathrm{id}$ is deduced from the lax-linear structure $(-) \otimes F(1_{\mathscr{V}}) \to F(- \otimes 1_{\mathscr{V}}) \simeq F(-)$. We check the zig-zag identities. The composite $\kappa \to \kappa\iota\kappa \to \kappa$ identifies (objectwise) with $F(1_{\mathscr{V}}) \simeq 1_{\mathscr{V}} \otimes F(1_{\mathscr{V}}) \to F(1_{\mathscr{V}} \otimes 1_{\mathscr{V}}) \simeq F(1_{\mathscr{V}})$ which is an equivalence because lax $\mathscr{V}$-linear functors satisfy a unitality axiom. It is easy to see that the composite $\iota \to \iota\kappa\iota \to \iota$ is homotopic to the identity. □

Corollary 2.2.19. *The functor κ in Lemma 2.2.18 acquires a canonical lax monoidal structure, for the composition monoidal structure on $\mathrm{Fun}^{\mathrm{lax}}_{\mathscr{V}}(\mathscr{V}, \mathscr{V})$. Further, there is an induced adjunction on algebra objects.*

Proof. It is clear that ι is strongly monoidal. Therefore κ acquires a canonical lax monoidal structure by [Lur17, Corollary 7.3.2.7] or [HHLN23, Theorem A]. □

This Lemma has an obvious oplax version.

Lemma 2.2.20. *There is an adjunction of ∞-categories*

$$\kappa' : \mathrm{Fun}^{\mathrm{oplax}}_{\mathscr{V}}(\mathscr{V}, \mathscr{V}) \leftrightarrows \mathscr{V} : \iota' \tag{2.81}$$

in which the right adjoint ι' sends $V \mapsto - \otimes V$ and the left adjoint κ' sends $F \mapsto F(1_{\mathscr{V}})$. (Note again here that the superscript oplax *stands for* oplax $\mathscr{V}$-linear *functors and* not *oplax monoidal functors). The functor κ' acquires a canonical oplax monoidal structure, for the composition monoidal structure on $\mathrm{Fun}^{\mathrm{oplax}}_{\mathscr{V}}(\mathscr{V}, \mathscr{V})$. Further, there is an induced adjunction on coalgebra objects.*

Now let $A \in \mathsf{Alg}(\mathscr{V})$ be an algebra object. Then $\mathrm{RMod}_A \mathscr{V}$ is a $\mathscr{V}$-linear category. We can formulate the following generalization of Lemma 2.2.18.

[17] I am grateful to Shay Ben-Moshe for explaining this to me.

Lemma 2.2.21. *Let $\mathcal{M}$ be a $\mathscr{V}$-linear category. There is an adjunction of ∞-categories*

$$\iota : \mathrm{LMod}_A \mathcal{M} \leftrightarrows \mathrm{Fun}^{\mathrm{lax}}_{\mathscr{V}}(\mathrm{RMod}_A \mathscr{V}, \mathcal{M}) : \kappa \tag{2.82}$$

in which the underlying object of $\kappa(F)$ is $F(A)$ and the left adjoint ι sends $M \mapsto (V \mapsto V \otimes_A M)$.

Proof. The functor κ is explicitly defined as follows. We first note that the construction of [Lur17, Remark 4.6.2.9] works just as well for lax-linear functors, so that there is a canonical functor

$$\mathrm{Fun}^{\mathrm{lax}}_{\mathscr{V}}(\mathrm{RMod}_A \mathscr{V}, \mathcal{M}) \to \mathrm{Fun}_{\mathscr{V}}(\mathrm{LMod}_A(\mathrm{RMod_A}\mathscr{V}), \mathrm{LMod}_A \mathcal{M}) \tag{2.83}$$

then evaluation on the object A (considered as a left and right A-module) gives the required functor κ. The unit morphism $\mathrm{id} \to \kappa\iota$ is the canonical equivalence $\mathrm{id} \simeq A \otimes_A \mathrm{id}$. The counit morphism $\kappa\iota \to \mathrm{id}$ is the composite

$$(-)\otimes_A F(A) \simeq \left|(-)\otimes A^{\otimes\bullet}\otimes F(A)\right| \to \left|F(-\otimes A^{\otimes\bullet+1})\right| \to F\big(|-\otimes A^{\otimes\bullet+1}|\big) \simeq F(-), \tag{2.84}$$

where we used the bar construction of [Lur17, §4.4.2] and lax linearity of F. It is easy to see that the composite $\iota \to \iota\kappa\iota \to \iota$ is homotopic to the identity. We check the composite $\kappa \to \kappa\iota\kappa \to \kappa$. This identifies with

$$F(A) \simeq A \otimes_A F(A) \simeq |A^{\otimes\bullet+1} \otimes F(A)| \to |F(A^{\otimes\bullet+2})| \simeq F(A). \tag{2.85}$$

The morphism $|A^{\otimes\bullet+1} \otimes F(A)| \to |F(A^{\otimes\bullet+2})|$ is equivalent to $1_{\mathscr{V}} \otimes F(A) \to F(1_{\mathscr{V}} \otimes A)$ which is an equivalence by the unitality property of lax $\mathscr{V}$-linear functors. □

Specialising to the situation of $\mathcal{M} = \mathrm{RMod}_A \mathscr{V}$ gives the following.

Corollary 2.2.22. *There is an adjunction of ∞-categories*

$$\iota : {}_A\mathrm{BMod}_A \mathscr{V} \leftrightarrows \mathrm{Fun}^{\mathrm{lax}}_{\mathscr{V}}(\mathrm{RMod}_A \mathscr{V}, \mathrm{RMod}_A \mathscr{V}) : \kappa \tag{2.86}$$

in which the left adjoint ι is strongly monoidal and the right adjoint κ is lax monoidal (for the convolution monoidal structure on bimodules and the composition monoidal structure on endofunctors). Further, there is an induced adjunction on algebra objects.

Proof. It is well-known that the functor ι is strongly monoidal. Hence κ acquires a canonical lax monoidal structure by [Lur17, Corollary 7.3.2.7] or [HHLN23, Theorem A]. □

It seems possible that there is an "oplax" version of Corollary 2.2.22 in which A is replaced with a coalgebra object, and one uses bicomodules instead of bimodules.

Example 2.2.23. *Let $\mathscr{C}$ be a $\mathscr{V}$-linear category, and let $F : \mathrm{RMod}_A \mathscr{V} \leftrightarrows \mathscr{C} : G$ be an adjunction in which the left adjoint F is $\mathscr{V}$-linear. Then by Theorem 2.2.17 the right adjoint G acquires a canonical lax $\mathscr{V}$-linear structure, so that the endofunctor GF of $\mathrm{RMod}_A \mathscr{V}$ acquires a canonical lax $\mathscr{V}$-linear structure. By Corollary 2.2.22 the object $GF(A)$ acquires structure of an A-A bimodule object equipped with a natural transformation*

$$GF(A) \otimes_A (-) \to GF(-) \tag{2.87}$$

of endofunctors of $\mathrm{RMod}_A \mathscr{V}$, coming from the counit $\iota\kappa \to \mathrm{id}$. Further, since GF is a monad (i.e., an algebra object in endofunctors), Corollary 2.2.22 implies that $GF(A)$ acquires the structure of an algebra object under convolution and the natural transformation (2.87) *is a morphism of monads.*

2.3 Theory of abstract six-functor formalisms

Definition 2.3.1. *[Man22, Definition A.5.1] A* geometric setup *is a pair* $(\mathscr{C}, E)$ *where* $\mathscr{C}$ *is an* ∞*-category and* E *is a collection of morphisms of* $\mathscr{C}$ *such that:*

- ⋆ *E contains all equivalences, is stable under composition, and pullbacks of morphisms in E by arbitrary morphisms of $\mathscr{C}$ exist and remain in E.*

Remark 2.3.2 (Important remark)**.** *Our definition of a geometric setup does not require E to satisfy the* right-cancellation property *(which says that E is closed under the formation of diagonals). In particular, our convention follows [Man22, §A.5] which is different to [HM24] in this way.*

Given a geometric setup one can define an ∞-category $\mathrm{Corr}(\mathscr{C}, E)$, which can be described informally as follows [Man22, §A.5].

- ⋆ The objects of $\mathrm{Corr}(\mathscr{C}, E)$ are the same as those of $\mathscr{C}$.
- ⋆ Morphisms $X \dashrightarrow Y$ of $\mathrm{Corr}(\mathscr{C}, E)$ are given by spans $X \xleftarrow{g} U \xrightarrow{f} Y$ with $f \in E$. The composite of $X \leftarrow U \to Y$ and $Y \leftarrow V \to Z$ is given by the composed span $X \leftarrow U \leftarrow U \times_Y V \to V \to Z$.
- ⋆ $\mathrm{Corr}(\mathscr{C}, E)$ has a monoidal structure built from the coCartesian monoidal structure on $\mathscr{C}^{\mathsf{op}}$.

A lax-monoidal functor $Q : \mathrm{Corr}(\mathscr{C}, E) \to \mathsf{Cat}_\infty$, (where the latter is endowed with the Cartesian monoidal structure), determines functors

$$\begin{aligned} g^* &:= Q(X \xleftarrow{g} Y = Y) : Q(X) \to Q(Y) \quad \text{and} \\ f_! &:= Q(X = X \xrightarrow{f} Y) : Q(X) \to Q(Y) \quad \text{and} \\ \otimes_X &: Q(X) \times Q(X) \to Q(X). \end{aligned} \tag{2.88}$$

Definition 2.3.3. *[LZ17, GR17, Man22].*

- ⋆ *A* six-functor formalism *is a lax-monoidal functor*[18] $Q : \mathrm{Corr}(\mathscr{C}, E) \to \mathsf{Cat}_\infty$ *such that g^* and $f_!$ admit right adjoints for every morphism g in $\mathscr{C}$ and every $f \in E$, and $M \otimes_X -$ admits a right adjoint for every $M \in Q(X)$.*
- ⋆ *The right adjoints are denoted g_*, $f^!$, and $\underline{\mathrm{Hom}}_X(M, -)$, respectively.*

Remark 2.3.4. *With notations as in Definition 2.3.3. The following basic identities are valid in any six-functor formalism.*

(i) (Projection formula). Let $[f : X \to Y] \in E$. There is a canonical equivalence

$$f_! \otimes_Y \mathrm{id} \simeq f_!(\mathrm{id} \otimes_X f^*) \tag{2.89}$$

of functors $Q(X) \times Q(Y) \to Q(Y)$.

(ii) Let $M \in Q(X)$ and $[f : X \to Y] \in E$. There is a canonical equivalence

$$f_* \underline{\mathrm{Hom}}_X(M, f^!(-)) \simeq \underline{\mathrm{Hom}}_Y(f_! M, -) \tag{2.90}$$

of functors $Q(Y) \to Q(Y)$. This follows by passing to adjoints in the projection formula (2.89).

[18] Where Cat_∞ is endowed with the Cartesian monoidal structure.

(iii) (Base-change). Suppose that we are given a Cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array} \tag{2.91}$$

with $f \in E$ (hence also $f' \in E$). Then:

(a) There is a canonical equivalence $g^* f_! \simeq f'_! g'^{,*}$ of functors $Q(X) \to Q(Y')$.

(b) There is a canonical equivalence $f^! g_* \simeq g'_* f'^{,!}$ of functors $Q(Y') \to Q(X)$. This follows by passing to right adjoints in the previous.

We may prefer to equivalently view a six-functor formalism as a map of operads $\mathrm{Corr}(\mathscr{C}, E) \to \mathsf{Cat}_\infty^\otimes$. Let $(\mathscr{C}, E)$ be a geometric setup and let $Q : \mathrm{Corr}(\mathscr{C}, E) \to \mathsf{Cat}_\infty^\otimes$ be a six-functor formalism. Let $\mathscr{C}_E$ be the subcategory of $\mathscr{C}$ where we only allow morphisms from E. There are functors $\mathscr{C}^{\mathsf{op}} \to \mathrm{Corr}(\mathscr{C}, E)$ and $\mathscr{C}_E \to \mathrm{Corr}(\mathscr{C}, E)$. On objects these are both induced by the identity; on morphisms the former sends $[g : X \to Y]$ to $[Y \xleftarrow{g} X = X]$ and the latter sends $[f : X \to Y] \in E$ to $[X = X \xrightarrow{f} Y]$. Via these functors we can restrict Q and obtain functors

$$Q^* : \mathscr{C}^{\mathsf{op}} \to \mathsf{Cat}_\infty \quad \text{and} \quad Q_! : \mathscr{C}_E \to \mathsf{Cat}_\infty. \tag{2.92}$$

By passing to right adjoints we obtain

$$Q_* : \mathscr{C} \to \mathsf{Cat}_\infty \quad \text{and} \quad Q^! : \mathscr{C}_E^{\mathsf{op}} \to \mathsf{Cat}_\infty. \tag{2.93}$$

In this context, we can make the following definition.

Definition 2.3.5. *[Sch22, Definition 4.14].*

(i) We say that a morphism $f : X \to Y$ in $\mathscr{C}$ is of $*$-descent *if the canonical morphism*

$$Q^*(Y) \to \lim_{[m] \in \Delta} Q^*(X^{m+1/Y}) \tag{2.94}$$

is an equivalence. We say that a morphism $f : X \to Y$ is of universal $*$-descent *if, for every $Z \in \mathscr{C}$ with a morphism $Z \to Y$, the base-change $X \times_Y Z \to Z$ is of $*$-descent.*

(ii) We say that a morphism $[f : X \to Y] \in E$ is of $!$-descent *if the canonical morphism*

$$Q^!(Y) \to \lim_{[m] \in \Delta} Q^!(X^{m+1/Y}) \tag{2.95}$$

is an equivalence. We say that a morphism $f : X \to Y$ is of universal $!$-descent *if, for every $Z \in \mathscr{C}$ with a morphism $Z \to Y$, the base-change $X \times_Y Z \to Z$ is of $!$-descent.*

Lemma 2.3.6 below can be viewed as the "easy direction" in a higher-categorical version of the Bénabou–Roubaud theorem. In the other direction, one has Lurie's Beck–Chevalley condition [Lur17, Corollary 4.7.5.3].

Lemma 2.3.6. *Fix a geometric setup $(\mathscr{C}, E)$ and a six-functor formalism $Q : \mathrm{Corr}(\mathscr{C}, E) \to \mathsf{Cat}_\infty^\otimes$.*

(i) Let $f : X \to Y$ be a morphism of $\mathscr{C}$ which is of $*$-descent, c.f. Definition 2.3.5(i). Then f^* induces an equivalence of categories $Q(Y) \simeq \mathrm{Comod}_{f^*f_*} Q(X)$, where the latter is the category of comodules over the comonad f^*f_*.

(ii) Let $f : X \to Y$ be a morphism of E which is of $!$-descent, c.f. Definition 2.3.5(ii). Then $f^!$ induces an equivalence of categories $Q(Y) \simeq \mathrm{Mod}_{f^!f_!} Q(X)$, where the latter is the category of comodules over the comonad $f^!f_!$.

Proof. (i): This is [Cam24, Proposition 3.1.27].

(ii): We adapt the argument of [Cam24, Proposition 3.1.27]. The proof is an application of the Barr–Beck–Lurie theorem [Lur17, Theorem 4.7.3.5]. The assumption that f satisfies !-descent implies that $f^!$ is conservative. It remains to show that $f^!$ preserves geometric realizations of $f^!$-split simplicial objects.

Let $(M_m)_{[m]\in\Delta_+^{\mathsf{op}}}$ be an augmented simplicial object of $Q(Y)$ which becomes split (i.e., acquires extra degeneracies), after applying $f^!$. For each $n \geqslant 0$ let $f_{n+1} : Y^{n+1/X} \to X$ be the projection.

Then, for all $n \geqslant 1$, the augmented simplicial object $(f^!_{n+1}M_m)_{m\in\Delta_+^{\mathsf{op}}}$ of $Q(Y^{n+1/X})$ is split. Let us set $N_{n+1} = \mathrm{colim}_{[m]\in\Delta^{\mathsf{op}}} f^!_{n+1}M_m$. Note that the existence of the splitting implies that the collection $(N_{n+1})_{[n]\in\Delta^{\mathsf{op}}}$ is a Cartesian section of $Q(Y^{\bullet+1/X})$. Now, we compute

$$\begin{aligned}\operatorname*{colim}_{[m]\in\Delta^{\mathsf{op}}} M_m &\simeq \operatorname*{colim}_{[m]\in\Delta^{\mathsf{op}}}\operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} f_{n+1,!}f^!_{n+1}M_m \\ &\simeq \operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} f_{n+1,!}\operatorname*{colim}_{[m]\in\Delta^{\mathsf{op}}} f^!_{n+1}M_m \\ &\simeq \operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} f_{n+1,!}N_{n+1}.\end{aligned} \tag{2.96}$$

Now, since $(N_{n+1})_{[n]\in\Delta^{\mathsf{op}}}$ is a Cartesian section, !-descent for f says that $f^!\,\mathrm{colim}_{[m]\in\Delta^{\mathsf{op}}} M_m = N_1 = \mathrm{colim}_{[m]\in\Delta^{\mathsf{op}}} f^!M_m$. Therefore $f^!$ preserves geometric realizations of $f^!$-split simplicial objects and we are done. □

2.3.1 An extension formalism for abstract six-functor formalisms

Remark 2.3.7. *The content of this subsection originally appeared in [Soo24] and later essentially the same result appeared as [HM24, Theorem 3.4.11]. We claim no originality for the results of this subsection: the main result (Theorem 2.3.10) is just a re-hashing of [Sch22, Theorem 4.20], and all of the extension principles were already developed by Mann [Man22, §A.5]. We still include it because our formulation is perhaps slightly closer to [Sch22, Theorem 4.20]: in particular we chose to keep the part about being "stable under disjoint unions" because we find it useful. This property appears to have something to do with idempotent completeness of the "representable objects", see the proof of Theorem 2.3.17.*

In this subsection we will fix two geometric setups $(\mathscr{C}, E_0)$ and $(\mathscr{C}_0, E_{00})$, and we will also fix a six-functor formalism

$$Q : \mathrm{Corr}(\mathscr{C}, E_0) \to \mathsf{Cat}_\infty^{\otimes}, \tag{2.97}$$

subject to the following list of Assumptions.

Assumptions 2.3.8. *(i) $\mathscr{C}$ admits all fiber products and all small coproducts. We will denote the initial object of $\mathscr{C}$ by $\emptyset$ and the final object by $*$.*

(ii) $\mathscr{C}_0 \subseteq \mathscr{C}$ *is a full subcategory stable under fiber products in* $\mathscr{C}$*.*

(iii) *For all* $X \in \mathscr{C}$*,* $Q(X)$ *is presentable.*

(iv) *A morphism* $f : X \to Y$ *of* $\mathscr{C}$ *belongs to* E_0 *if and only if for every* $Z \in \mathscr{C}_0$ *with a morphism* $Z \to Y$*, the base-change* $[X \times_Y Z \to Z] \in E_{00}$*.*

(v) *For all* $X \in \mathscr{C}$*, the canonical morphism*

$$Q^*(X) \to \lim_{Y \in \mathscr{C}_{0}{}^{\mathsf{op}}_{/X}} Q^*(Y) \tag{2.98}$$

is an equivalence. That is, $Q^* : \mathscr{C}^{\mathsf{op}} \to \mathsf{Cat}_\infty$ *is the right Kan extension of* $Q^*|_{\mathscr{C}_0^{\mathsf{op}}} \to \mathsf{Cat}_\infty$ *along* $\mathscr{C}_0^{\mathsf{op}} \to \mathscr{C}^{\mathsf{op}}$*.*

(vi) *Coproducts in* $\mathscr{C}$ *are* disjoint *and* universal*. This means that:*

(a) *(Disjoint). For all* $X, Y \in \mathscr{C}$*, the morphism* $\emptyset \to X \times_{X \coprod Y} Y$ *is an equivalence.*

(b) *(Universal). For all small families* $\{X_i \to Y\}_{i \in \mathcal{I}}$ *of morphisms in* $\mathscr{C}$ *and any morphism* $Z \to Y$ *in* $\mathscr{C}$*, the canonical morphism*

$$\coprod_{i \in \mathcal{I}} (X_i \times_Y Z) \to \Big(\coprod_{i \in \mathcal{I}} X_i\Big) \times_Y Z \tag{2.99}$$

is an equivalence.

(vii) *For all* $X, Y \in \mathscr{C}$ *the morphism* $X \to X \coprod Y \in E_0$*.*

(viii) $Q^! : \mathscr{C}_{E_0}^{\mathsf{op}} \to \mathsf{Cat}_\infty$ *preserves small products. That is, for all small families* $\{X_i\}_{i \in \mathcal{I}}$ *of objects of* $\mathscr{C}$*, the natural morphism* $Q^!\big(\coprod_{i \in \mathcal{I}} X_i\big) \to \prod_{i \in \mathcal{I}} Q^!(X_i)$ *is an equivalence. This condition makes sense by (vii).*

(ix) *Let* δE_0 *be the class of morphisms in* $\mathscr{C}$ *whose diagonal belongs to* E_0*. Then we require that* $E_0 \subseteq \delta E_0$[19]*.*

Definition 2.3.9. *[Sch22, Definition 4.18] Let* $(\mathscr{C}, E)$ *be another geometric setup such that* $E_0 \subseteq E$*.*

(i) *We say that the class* E *is* stable under disjoint unions *if for all small families* $\{X_i \to Y\}_{i \in \mathcal{I}}$ *of morphisms of* E*, the morphism* $\coprod_{i \in \mathcal{I}} X_i \to Y$ *belongs to* E*.*

(ii) *We say that the class* E *is* $*$-local on the target *if whenever* $f : X \to Y$ *is a morphism of* $\mathscr{C}$ *such that, for all* $Z \in \mathscr{C}_0$ *with a map* $Z \to Y$*, the base change* $X \times_Y Z \to Z$ *belongs to* E*, then* $f \in E$*.*

(iii) *We say that* Q extends uniquely to $(\mathscr{C}, E)$ *if:*

(a) *There exists a six-functor formalism* $Q' : \mathrm{Corr}(\mathscr{C}, E) \to \mathsf{Cat}_\infty^{\otimes}$ *equipped with an equivalence* $Q'|_{\mathrm{Corr}(\mathscr{C}, E_0)} \simeq Q$*.*

[19] Note that this differs slightly to [Sch22, Definition 4.18](3). I think that this change is actually necessary in order to run the argument of [Sch22, Theorem 4.20] since, without it, the hypothesis (d) in [Man22, Proposition A.5.14] might not be satisfied.

(b) *Whenever* $Q'' : \mathrm{Corr}(\mathscr{C}, E) \to \mathsf{Cat}_\infty^\otimes$ *is another six-functor formalism whose restriction to* $\mathrm{Corr}(\mathscr{C}, E_0)$ *is equipped with an equivalence*

$$Q'|_{\mathrm{Corr}(\mathscr{C}, E_0)} \simeq Q''|_{\mathrm{Corr}(\mathscr{C}, E_0)} \tag{2.100}$$

then there exists a unique (up to contractible choice) equivalence $Q' \simeq Q''$ *lifting* (2.100).

(iv) *Assume that* Q *extends uniquely to* $(\mathscr{C}, E)$*. We say that* E *is* !-local on the source *if whenever* $f : X \to Y$ *is a morphism in* $\mathscr{C}$ *such that there exists* $[g : X' \to X] \in E$ *of universal* !*-descent such that* $fg \in E$*, then* $f \in E$*.*

(v) *Assume that* Q *extends uniquely to* $(\mathscr{C}, E)$*. We say that* E *is* tame *if whenever* $Y \in \mathscr{C}_0$ *and* $[f : X \to Y] \in E$ *then there exists a small family* $\{X_i \to Y\}_{i \in \mathcal{I}}$ *of morphisms in* E_{00} *and a morphism* $\left[\coprod_{i \in \mathcal{I}} X_i \to X\right] \in E$ *over* Y *which is of universal* !*-descent.*

Theorem 2.3.10. *[Sch22, Theorem 4.20] With notations as introduced in this subsection. Under the Assumptions 2.3.8, there is a (minimal) collection of morphisms* $E \supseteq E_0$ *of* $\mathscr{C}$ *such that* Q *extends uniquely to* $(\mathscr{C}, E)$ *and* E *is stable under disjoint unions,* $*$*-local on the target, is* !*-local on the source, is tame, and satisfies* $E \subseteq \delta E$*.*

Proof. This is the same as [Sch22, Theorem 4.20]. We reproduce the proof here for convenience, and also to convince the reader that the result is true in our slightly more general context.

Let A be the class of all classes E of morphisms in $\mathscr{C}$ such that $(\mathscr{C}, E)$ is a geometric setup, Q extends uniquely to $(\mathscr{C}, E)$, and E is tame and satisfies $E \subseteq \delta E$. By Assumptions 2.3.8(iv) and 2.3.8(ix), we have $E_0 \in A$, and this is minimal.

In the proof of *loc. cit.* it was observed that filtered unions can be taken in the class A, and that this is a convenient way to organise the fact that we will have to iterate the extension principles of [Man22, §A.5] transfinitely many times. We will proceed in steps. The steps will be indexed by ordinals starting at 2.

Step 2: Let $E \in A$. We let E' be the collection of morphisms which can be written as $\coprod_i X_i \to Y$ such that each $[X_i \to Y] \in E$. By Assumptions 2.3.8(iii), 2.3.8(vi), 2.3.8(vii), and 2.3.8(viii), we may apply [Man22, Proposition A.5.12] to deduce that $(\mathscr{C}, E')$ is a geometric setup and Q extends uniquely to $(\mathscr{C}, E')$. Due to Assumptions 2.3.8(vi)(b) and 2.3.8(viii), the class E' is again tame. Using Assumptions 2.3.8(vi) and (vii) we see that $E' \subseteq \delta E'$. The class E' is clearly stable under disjoint unions. We conclude that any $E \in A$ can be minimally enlarged to $E' \in A$ which is stable under disjoint unions.

Step 3: Again, let $E \in A$. Let E' be the class of morphisms $f : Y \to X$ in $\mathscr{C}$ such that there exists $[g : Z \to Y] \in E$ of universal !-descent such that $fg \in E$. By Lemma 2.3.11 below, $(\mathscr{C}, E')$ is again a geometric setup, and E' is tame and satisfies $E' \subseteq \delta E'$. Now we can apply [Man22, Proposition A.5.14] to extend Q uniquely to $(\mathscr{C}, E')$, taking the class S of covers in *loc. cit.* to be those of universal !-descent. We note in particular that the assumption (a) of *loc. cit.* is satisfied due to Assumption 2.3.8(iii) and the assumption (d) of *loc. cit.* is satisfied since $E \subseteq \delta E$. We conclude that any $E \in A$ can be minimally enlarged to $E' \in A$ which is !-local on the source.

Step ω: By setting $E_1 := E_0$ and alternately applying Steps 2 and 3, we obtain a chain

$$E_1 \subseteq E_2 \subseteq E_3 \subseteq E_4 \subseteq \dots \tag{2.101}$$

such that for even $n \geqslant 2$, E_n is stable under disjoint unions and for odd $n \geqslant 3$, E_n is !-local on the source. One then sets

$$E_\omega := \bigcup_{n \geqslant 1} E_n \in A. \tag{2.102}$$

Then E_ω is stable under disjoint unions and !-local on the source.

Step $\omega + 1$: Now assume that we are given $E \in A$ which is stable under disjoint unions and !-local on the source. We let E' be the collection of morphisms $[f : X \to Y]$ of $\mathscr{C}$ such that for all $Z \in \mathscr{C}_0$ with a map $Z \to Y$, the pullback $[X \times_Y Z \to Z] \in E$. By Lemma 2.3.12 below, $(\mathscr{C}, E')$ is again a geometric setup, E' is tame and $E' \subseteq \delta E'$. Now one defines $\mathscr{C}' \subseteq \mathscr{C}$ to be the full subcategory of $\mathscr{C}$ on objects X which admit an E-morphism to an object of $\mathscr{C}_0$ and one sets E'' to be the restriction of E to $\mathscr{C}'$. It is not hard to see that the inclusion $\mathscr{C}' \subseteq \mathscr{C}$ preserves pullbacks of edges in E'' and that E' consists precisely of those morphisms whose pullback to $\mathscr{C}'$ belongs to E''. Therefore one may restrict Q to $(\mathscr{C}', E')$ and then use [Man22, Proposition A.5.16] to extend Q back to $(\mathscr{C}, E')$. In order to satisfy the uniqueness assumption in *loc. cit.* and hence conclude that this is the unique extension of Q from $(\mathscr{C}, E)$ to $(\mathscr{C}, E')$, we need to see that for each $X \in \mathscr{C}$, the canonical morphism

$$Q^*(X) \to \lim_{Y' \in (\mathscr{C}'_{/X})^{\mathrm{op}}} Q^*(Y') \tag{2.103}$$

is an equivalence, i.e., that Q^* is right Kan extended from $\mathscr{C}'$. This is a consequence of Assumption 2.3.8(v) and the transitivity of Kan extensions [Lur18a, Tag 0314].

Step $\omega \cdot 2$: The class $E_{\omega+1}$ may no longer be !-local on the source or stable under disjoint unions. Therefore one iterates Steps 2 and 3 again to obtain a chain $E_{\omega+1} \subseteq E_{\omega+2} \subseteq E_{\omega+3} \subseteq \dots$ such that $E_{\omega+2n}$ is stable under disjoint unions and $E_{\omega+2n+1}$ is !-local on the source, for all $n \geqslant 1$. One then sets $E_{\omega\cdot 2} := \bigcup_{n \geqslant 1} E_{\omega+n} \in A$, which is stable under disjoint unions and !-local on the source.

Step ω^2: Continuing in this way, we obtain an increasing sequence $\{E_\alpha\}_{\alpha<\omega^2}$ of classes in A. For each $m \geqslant 1$, $E_{\omega\cdot m+1}$ is $*$-local on the target and $E_{\omega\cdot m}$ is !-local on the source and stable under disjoint unions. Therefore $E_{\omega^2} := \bigcup_{\alpha<\omega^2} E_\alpha \in A$ is stable under disjoint unions, $*$-local on the target, and !-local on the source. ☐

The following auxiliary Lemmas were used in the proof of Theorem 2.3.10.

Lemma 2.3.11. *With notations as introduced in this subsection. Let $(\mathscr{C}, E)$ be a geometric setup with $E_0 \subseteq E$ and assume that Q extends uniquely to $(\mathscr{C}, E)$. Let E' be the class of morphisms $f : Y \to X$ in $\mathscr{C}$ such that there exists $[g : Z \to Y] \in E$ of universal !-descent such that $fg \in E$. Then:*

(i) E' is stable under base-change.

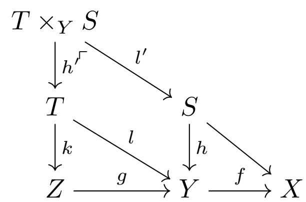
(ii) E' is stable under composition.

(iii) If E satisfies the right cancellation property (that is, $E \subseteq \delta E$), then so does E'.

(iv) If E is tame then so is E'.

Proof. (i): This is clear.

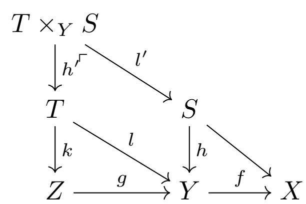
(ii): Let $[f : Y \to X]$ and $[g : Z \to Y] \in E'$, so that there exists $[h : S \to Y]$ and $[k : T \to Z] \in E$ of universal !-descent such that gk and $fh \in E$. Consider the diagram



(2.104)

to be sure, in this diagram the parallelogram is Cartesian and we have written $l := gk$. By base change $l' \in E$ and so $gkh' = hl' \in E$. Also $kh' \in E$ is of universal !-descent because this class is stable under composition and base-change.

(iii): Let $[f : Y \to X]$ and $[g : Z \to Y]$ be morphisms of $\mathscr{C}$ such that fg and $f \in E'$, so that there exists $[h : S \to Y]$ and $[k : T \to Z]$ of universal !-descent such that fgk and $fh \in E$. Consider again the diagram



(2.105)

one has $fgkh' = fhl' \in E$. Therefore, by the right cancellation property for E, $l' \in E$. Therefore $gkh' = hl' \in E$. Also $kh' \in E$ is of universal !-descent. Therefore $g \in E'$.

(iv): This follows by unravelling the definitions. Let $[f : Y \to X] \in E'$ be such that $Y \in \mathscr{C}_0$, so that there exists $[g : Z \to Y] \in E$ of universal !-descent such that $fg \in E$. By tameness of E there exists a small family $\{X_i \to X\}_{i \in \mathcal{I}}$ of morphisms of E_{00} and a morphism $\coprod_{i \in \mathcal{I}} X_i \to Z$ over X which is of universal !-descent. The composite $\coprod_{i \in \mathcal{I}} X_i \to Z \to X$ is then a morphism of universal !-descent over X. Therefore E' is tame. ☐

Lemma 2.3.12. *With notations as introduced in this subsection. Let $(\mathscr{C}, E)$ be a geometric setup with $E_0 \subseteq E$ and assume that Q extends uniquely to $(\mathscr{C}, E)$. Let E' be the collection of morphisms $[f : X \to Y]$ of $\mathscr{C}$ such that for all $Z \in \mathscr{C}_0$ with a map $Z \to Y$, the pullback $[X \times_Y Z \to Z] \in E$. Then:*

(i) E' is stable under base-change.

(ii) If E is tame then E' is tame.

(iii) If E satisfies the right cancellation property (that is, $E \subseteq \delta E$), then so does E'.

(iv) If E is stable under disjoint unions, !-local on the source, and tame, then E' is stable under composition.

Proof. (i): This is clear.

(ii): This is clear since, by the definition, if $[f : X \to Y] \in E'$ has $Y \in \mathscr{C}_0$ then $f \in E$.

(iii): This is clear.

(iv): Let $[f : Y \to X]$ and $[g : Z \to Y] \in E'$, let $[S \to X]$ be a morphism from an object $S \in \mathscr{C}_0$ and set $T := Y \times_X S$, $R := Z \times_X S$ and let f' and g' be the pullbacks of f and g. By tameness there exists a small family $\{S_i \to S\}_{i \in \mathcal{I}}$ of morphisms of E_{00} and a morphism $\coprod_{i\in\mathcal{I}} S_i \to T$ over S of universal !-descent. We can summarise this information in the diagram

$$\begin{array}{ccccc} \coprod_{i\in\mathcal{I}}(S_i \times_T R) & \longrightarrow & \coprod_{i\in\mathcal{I}} S_i & & \\ \downarrow & & \downarrow & \searrow & \\ R & \xrightarrow{g'} & T & \xrightarrow{f'} & S \\ \downarrow & & \downarrow & & \downarrow \\ Z & \xrightarrow{g} & Y & \xrightarrow{f} & X \end{array} \tag{2.106}$$

where we used Assumption 2.3.8(vi)(b). Since the class E is !-local on the source and stable under disjoint unions, to check that $g' \in E$ it suffices to check that each $[S_i \times_T R \to S_i] \in E$. But this is true by the assumption that $g \in E'$. Therefore E' is stable under composition. □

2.3.2 The six-functor formalism of relative algebraic geometry

In this section I will assume that

- ⋆ $\mathscr{V}$ is a stable presentably symmetric monoidal ∞-category. We write $\otimes$ for the monoidal structure on $\mathscr{V}$.

We define $\mathscr{E} := \mathsf{CAlg}(\mathscr{V})^{\mathsf{op}}$. We use the formal expression $\mathrm{Spec}(A)$ to denote the object of $\mathscr{E}$ corresponding to $A \in \mathsf{CAlg}(\mathscr{V})$. By [Lur17, Theorem 4.5.3.1], (see also [Lur17, Remark 4.5.3.2]), there is a functor

$$\mathrm{QCoh} : \mathscr{E}^{\mathsf{op}} \to \mathsf{CAlg}(\mathsf{Pr}^L_{\mathsf{st}}), \tag{2.107}$$

which is given on objects by

$$\mathrm{QCoh}(\mathrm{Spec}(A)) := \mathrm{Mod}_A \mathscr{V}, \tag{2.108}$$

and sends $f : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ to $f^* = A \otimes_B -$. The functor f^* admits a right adjoint f_* which is nothing but the forgetful functor at the level of modules.

Proposition 2.3.13. *The functor* $\mathrm{QCoh} : \mathscr{E}^{\mathsf{op}} \to \mathsf{CAlg}(\mathsf{Pr}^L_{\mathsf{st}})$ *extends to a six-functor formalism*

$$\mathrm{QCoh} : \mathrm{Corr}(\mathscr{E}, \mathrm{all})^{\otimes} \to \mathsf{Pr}^{L,\otimes}_{\mathsf{st}} \tag{2.109}$$

such that for every morphism f *in* $\mathscr{E}$ *one has* $f_* = f_!$.

Proof. This is a straightforward application of [Man22, Proposition A.5.10] applied to the decomposition $(I, P) = (\text{equivalences}, \text{all})$. Base change and the projection formula both follow from standard associativity properties of $\otimes$. □

Notations 2.3.14.

- ⋆ *Let* $\mathsf{Aff} \subseteq \mathscr{E}$ *be a full subcategory stable under fiber products and retracts.*
- ⋆ *Let* τ *be a Grothendieck topology on* Aff *such that* QCoh^* *is a sheaf in this topology. (By evaluation on the unit object, this in particular implies that* τ *is subcanonical).*
- ⋆ *Let* $\mathsf{Stk} := \mathrm{Shv}_\tau(\mathsf{Aff}, \infty\mathsf{Grpd})$ *equipped with its natural topology as a topos, that is, the topology of effective epimorphisms.*
- ⋆ *Let* rep *be the collection of morphisms in* Stk *which are representable in* Aff.

The Yoneda embedding induces a morphism of geometric setups $(\mathsf{Aff}, \mathrm{all}) \to (\mathsf{Stk}, \mathrm{rep})$. By [Man22, Proposition A.5.16] QCoh extends to a six-functor formalism

$$\mathrm{QCoh} : \mathrm{Corr}(\mathsf{Stk}, \mathrm{rep})^{\otimes} \to \mathsf{Pr}_{\mathsf{st}}^{L,\otimes}. \tag{2.110}$$

such that QCoh^* is a sheaf in the effective-epimorphism topology.

Lemma 2.3.15. *Let $g : Y \to X \in \mathrm{rep}$. Then g_* satisfies base-change and the projection formula, and further g_* is conservative.*

Proof. Using that τ is subcanonical, all statements follow from descent. □

Corollary 2.3.16. *In the six-functor formalism* (2.110)*, for every morphism $[g : Y \to X] \in \mathrm{rep}$, there is a natural equivalence $g_! \xrightarrow{\sim} g_*$.*

Proof. By a straightforward application of [Man22, Proposition A.5.10], using Lemma 2.3.15 above, one constructs a second six-functor formalism on $(\mathsf{Stk}, \mathrm{rep})$, such that for every $[g : Y \to X] \in \mathrm{rep}$ one has $g_! = g_*$. Now the unicity assertion in [Man22, Proposition A.5.16] implies that this six-functor formalism is equivalent to the one constructed in (2.110) above, which gives the Corollary. □

Now we apply the *extension formalism* of §2.3.1.

Theorem 2.3.17 (The six-functor formalism of relative algebraic geometry)**.** *There exists a (minimal) class of edges $E \supseteq \mathrm{rep}$ of Stk such that* QCoh *extends to a six-functor formalism on (Stk, E) and E is stable under disjoint unions, $*$-local on the target, !-local on the source, is tame, and satisfies $E \subseteq \delta E$.*

Proof. We will apply Theorem 2.3.10 with

$$(\mathscr{C}_0, E_{00}) := (\mathsf{Aff}, \mathrm{all}), \quad \text{and} \quad (\mathscr{C}, E_0) := (\mathsf{Stk}, \mathrm{rep}). \tag{2.111}$$

We need to check that the Assumptions 2.3.8(i)-(ix) are satisfied. The Assumptions 2.3.8(i), (vi) are satisfied in any ∞-topos. The Assumption 2.3.8(v) follows from the fact that QCoh^* is left Kan extended along the Yoneda embedding, c.f. [Man22, Proposition A.5.16], and then 2.3.8(iii) follows because $\mathrm{QCoh}(Y)$ is presentable for every $Y \in \mathsf{Aff}$. The Assumption 2.3.8(ii) follows since the topology τ is subcanonical. Assumption 2.3.8(iv) is clear from the definition of a representable morphism and 2.3.8(ix) is also easily verified. So the only things to really check are Assumptions 2.3.8(vii) and 2.3.8(viii).

For 2.3.8(vii), by base-change it suffices to show that the morphism $i : * \to * \coprod *$ into the first factor is representable. We note that this has a retraction r (the fold map) and in fact one can[20] write $* = \lim_{\mathrm{Idem}} (* \coprod *)$ where the ∞-category Idem is from [Lur09b, §4.4.5]. Now fiber products commute with limits so given $\mathrm{Spec}(A) \to * \coprod *$ we can write $* \times_{* \coprod *} \mathrm{Spec}(A) = \lim_{\mathrm{Idem}} \mathrm{Spec}(\mathrm{A})$. Using that the Yoneda embedding is fully-faithful (since τ is subcanonical) and preserves all limits which exist in Aff, we deduce that this is representable, since Aff is idempotent complete (by assumption, it is stable under retracts in $\mathscr{E}$).

For Assumption 2.3.8(viii) let us take objects $X, Y \in \mathsf{Stk}$ and consider the morphisms $s : X \to X \coprod Y$ and $t : Y \to X \coprod Y$. By the previous these both belong to the class rep, so that in particular one has base-change for s_* and t_* against the upper-shriek functors.

[20] I would like to thank Sam Moore for a helpful discussion, and especially for explaining "Idem" to me.

Because QCoh* is a sheaf, one has $\mathrm{id} \simeq s_* s^* \oplus t_* t^*$. Now applying $s^!$ and using base-change we deduce that $s^! \simeq s^*$.

Now suppose we are given a small collection $\{X_i\}_{i \in \mathcal{I}}$ of objects of Stk and consider the morphisms $s_i : X_i \to \coprod_i X_i$ and $t_i : \coprod_{j \neq i} X_j \to \coprod_i X_i$. By the above argument we have $s_i^! \simeq s_i^*$. Since QCoh* is a sheaf, one has an equivalence $\prod_i s_i^* : \prod_i \mathrm{QCoh}(X_i) \xrightarrow{\sim} \mathrm{QCoh}(\coprod_i X_i)$. Hence also $\prod_i s_i^! : \prod_i \mathrm{QCoh}(X_i) \xrightarrow{\sim} \mathrm{QCoh}(\coprod_i X_i)$ is an equivalence, as required. □

Remark 2.3.18. *It is my hope that the discussion about idempotent-completeness in the above Proof can be helpful to the authors of [HM24], in particular in relation to the discussion above [HM24, Lemma 3.4.13].*

2.3.3 A digression on Fourier–Mukai transforms and the tensor product formula

I am grateful to Peter Scholze for explaining the main ideas of this subsection to me during a conversation at the Clay Math conference in October 2024. In this subsection we use notations as in §2.3.2. We let E be the collection of edges in Stk coming from Theorem 2.3.17 and let

$$\mathrm{QCoh} : \mathrm{Corr}(\mathsf{Stk}, E)^{\otimes} \to \mathsf{Pr}^{L,\otimes}_{\mathsf{st}}. \tag{2.112}$$

be the six-functor formalism coming from Theorem 2.3.17. Let us fix a base $Y \in \mathsf{Stk}$. Let $(\mathsf{Stk}_E)_{/Y}$ be the full subcategory of $\mathsf{Stk}_{/Y}$ consisting of those objects whose morphism to Y belongs to the class E. Because the class E has the right cancellation property then $((\mathsf{Stk}_E)_{/Y}, \mathrm{all})$ is a geometric setup and $((\mathsf{Stk}_E)_{/Y}, \mathrm{all}) \to (\mathsf{Stk}, E)$ is a morphism of geometric setups and hence induces a morphism on the respective categories of correspondences. By precomposition with (2.112) we obtain a six-functor formalism

$$\mathrm{QCoh} : \mathrm{Corr}((\mathsf{Stk}_E)_{/Y}, \mathrm{all})^{\otimes} \to \mathsf{Pr}^{L,\otimes}_{\mathsf{st}} \tag{2.113}$$

which by [HM24, Lemma 3.2.5] factors over $\mathrm{Mod}_{\mathrm{QCoh}(Y)} \mathsf{Pr}^{L,\otimes}_{\mathsf{st}}$. That is, we obtain a functor of operads

$$\mathrm{QCoh} : \mathrm{Corr}((\mathsf{Stk}_E)_{/Y}, \mathrm{all})^{\otimes} \to \mathrm{Mod}_{\mathrm{QCoh}(Y)} \mathsf{Pr}^{L,\otimes}_{\mathsf{st}}. \tag{2.114}$$

By transferring [HM24, Definition C.3.1] the self-enrichment [HM24, §2.4] of

$$\mathrm{Corr}((\mathsf{Stk}_E)_{/Y}, \mathrm{all})^{\otimes} \tag{2.115}$$

along this morphism of operads we obtain the *category of correspondences* $\mathcal{K}_{\mathrm{QCoh},Y}$, which by [HM24, Lemma C.3.7] is equipped with a $\mathrm{Mod}_{\mathrm{QCoh}(Y)} \mathsf{Pr}^{L,\otimes}_{\mathsf{st}}$-enriched functor

$$\Psi_Y : \mathcal{K}_{\mathrm{QCoh},Y} \to \mathrm{Mod}_{\mathrm{QCoh}(Y)} \mathsf{Pr}^{L,\otimes}_{\mathsf{st}}. \tag{2.116}$$

In particular for any pair of morphisms $X \to Y \leftarrow Z$ in Stk both belonging to E, we obtain a "Fourier–Mukai transform":

$$FM : \mathrm{QCoh}(Z \times_Y X) \to \mathrm{Fun}^L_{\mathrm{QCoh}(Y)}(\mathrm{QCoh}(\mathrm{Z}), \mathrm{QCoh}(X)) \tag{2.117}$$

which, since Ψ_Y is enriched, is compatible with composition (i.e., takes convolution of kernels to composition of functors).

Definition 2.3.19. *We say that $X \in \mathsf{Stk}$ has* affine diagonal *if the diagonal $X \to X \times X$ belongs to the class* rep *of representable morphisms.*

Proposition 2.3.20. *Let $Y \in \mathsf{Stk}$ have affine diagonal. Let $f : \mathrm{Spec}(B) \to Y$ be an object of Aff mapping to Y and let $X \to Y$ be an arbitrary morphism in Stk. Then the natural morphism*

$$\mathrm{QCoh}(X) \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(\mathrm{Spec}(B)) \to \mathrm{QCoh}(X \times_Y \mathrm{Spec}(B)) \tag{2.118}$$

is an equivalence.

Proof. The proof of [BZFN10, Proposition 4.13] works in this generality. □

Definition 2.3.21. *We say a morphism $f : Z \to Y$ in Stk is* transformable *if it has affine diagonal, and for every affine $\mathrm{Spec}(B) \to Y$, there is a morphism $\mathrm{Spec}(A) \to Z \times_Y \mathrm{Spec}(B)$ from an affine, which is of universal !-descent.*

Theorem 2.3.22. *Let $Y \in \mathsf{Stk}$ have affine diagonal and let $f : Z \to Y$ be transformable. Then for any morphism $X \to Y$ in Stk the Fourier-Mukai transform*

$$FM : \mathrm{QCoh}(Z \times_Y X) \to \mathrm{Fun}^L_{\mathrm{QCoh}(Y)}(\mathrm{QCoh}(Z), \mathrm{QCoh}(X)), \tag{2.119}$$

is an equivalence.

We first prove an intermediate Lemma.

Lemma 2.3.23. *Let $Z, Y \in \mathsf{Stk}$ have affine diagonal, and assume that there exists a morphism $Z' = \mathrm{Spec}(A) \to Z$ from an affine, which is of universal !-descent. Then:*

(i) For any morphism $X \to Y$ in Stk, the natural morphism

$$\mathrm{QCoh}(X) \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(Z) \to \mathrm{QCoh}(X \times_Y Z) \tag{2.120}$$

is an equivalence.

(ii) $\mathrm{QCoh}(X)$ is dualizable and canonically self-dual as a $\mathrm{QCoh}(Y)$-module.

(iii) For any morphism $X \to Y$ in Stk the Fourier-Mukai transform

$$FM : \mathrm{QCoh}(Z \times_Y X) \to \mathrm{Fun}^L_{\mathrm{QCoh}(Y)}(\mathrm{QCoh}(Z), \mathrm{QCoh}(X)), \tag{2.121}$$

is an equivalence.

Proof. (i): Since f is assumed to be transformable, there exists a morphism $g : Z' = \mathrm{Spec}(A) \to Z$ from an affine, which is of universal !-descent. That is, the canonical morphism $\mathrm{QCoh}(Z) \to \lim_{[n]\in\Delta} \mathrm{QCoh}^!(Z'^{,n+1/Z})$ is an equivalence in Cat_∞, or, equivalently, the canonical morphism $\mathrm{colim}_{[n]\in\Delta^{\mathsf{op}}} \mathrm{QCoh}_!(Z'^{,n+1/Z}) \to \mathrm{QCoh}(Z)$ is an equivalence in $\mathrm{Mod}_{\mathrm{QCoh}(Y)} \mathsf{Pr}^L_{\mathsf{st}}$, (and similarly for any base-change of g over Y). Because the Lurie tensor product commutes with colimits separately in each variable, we therefore obtain

$$\begin{aligned}\mathrm{QCoh}(X) \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(Z) &\simeq \mathrm{QCoh}(X) \otimes_{\mathrm{QCoh}(Y)} \left(\operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} \mathrm{QCoh}_!\left(Z'^{,n+1/Z}\right)\right)\\ &\simeq \operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} \Big(\mathrm{QCoh}(X) \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}_!\left(Z'^{,n+1/Z}\right)\Big)\\ &\simeq \operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} \mathrm{QCoh}_!\left(X \times_Y Z'^{,n+1/Z}\right)\\ &\simeq \mathrm{QCoh}(X \times_Y Z),\end{aligned} \tag{2.122}$$

where in the third line we used Proposition 2.3.20, noting that all the iterated fiber products $Z'^{,n+1/Z}$ are affine because Z is assumed to have affine diagonal.

(ii): We first claim that $Z \to Y \in E$. Because the class E of !-able morphisms is !-local on the source, morphism $f : Z \to Y$ belongs to the class E if and only if $Z' \to Y$ does. But, since Y has affine diagonal, this morphism belongs to rep $\subseteq E$. We next claim that Z, viewed as an object of the symmetric-monoidal category

$$\mathrm{Corr}((\mathsf{Stk}_E)_{/Y}, \mathrm{all})^{\otimes}, \tag{2.123}$$

is dualizable and canonically self-dual. Indeed, the unit and counit and given by the correspondence

$$Y \xleftarrow{f} Z \xrightarrow{\Delta_f} Z \times_Y Z \tag{2.124}$$

and its opposite, respectively. Applying QCoh and using part (i), one obtains a unit

$$\mathrm{QCoh}(Y) \xrightarrow{\Delta_{f,!}f^*} \mathrm{QCoh}(Z \times_Y Z) \simeq \mathrm{QCoh}(Z) \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(Z) \tag{2.125}$$

and a counit

$$\mathrm{QCoh}(Z) \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(Z) \simeq \mathrm{QCoh}(Z \times_Y Z) \xrightarrow{f_! \Delta_f^*} \mathrm{QCoh}(Y), \tag{2.126}$$

which, using part (i), satisfy the zig-zag identities. This proves (ii).

(iii): Follows by combining (i) and (ii). □

Proof of Theorem 2.3.22. We follow [BZFN10, Theorem 4.14]. We first note that every transformable morphism belongs to the class E. Indeed, since the class E is $*$-local on the target, we reduce immediately to the affine case. Then the claim follows since E is !-local on the source.

Because the underlying site is subcanonical, we may write $Y \simeq \operatorname{colim}_{\mathrm{Spec}(B) \to Y} \mathrm{Spec}(B)$ as the colimit over all affines mapping to it. Then by $*$-descent one has

$$\begin{aligned} \mathrm{QCoh}(Z \times_Y X) &\simeq \lim \mathrm{QCoh}(Z \times_Y X \times_Y \mathrm{Spec}(B)) \\ &\simeq \lim \mathrm{QCoh}((Z \times_Y \mathrm{Spec}(B)) \times_{\mathrm{Spec}(B)} (X \times_Y \mathrm{Spec}(B))). \end{aligned} \tag{2.127}$$

By Lemma 2.3.23(iii) there are equivalences

$$\begin{aligned} &\mathrm{QCoh}((Z \times_Y \mathrm{Spec}(B)) \times_{\mathrm{Spec}(B)} (X \times_Y \mathrm{Spec}(B))) \\ \simeq\ &\mathrm{Fun}^L_{\mathrm{QCoh}(\mathrm{Spec}(B))}(\mathrm{QCoh}(Z \times_Y \mathrm{Spec}(B)), \mathrm{QCoh}(X \times_Y \mathrm{Spec}(B))) \\ \simeq\ &\mathrm{Fun}^L_{\mathrm{QCoh}(\mathrm{Spec}(B))}(\mathrm{QCoh}(Z) \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(\mathrm{Spec}(B)), \mathrm{QCoh}(X \times_Y \mathrm{Spec}(B))) \\ \simeq\ &\mathrm{Fun}^L_{\mathrm{QCoh}(Y)}(\mathrm{QCoh}(Z), \mathrm{QCoh}(X \times_Y \mathrm{Spec}(B))), \end{aligned} \tag{2.128}$$

where in the last line we used tensor-hom adjunction. Hence,

$$\begin{aligned} \mathrm{QCoh}(Z \times_Y X) &\simeq \lim \mathrm{Fun}^L_{\mathrm{QCoh}(Y)}(\mathrm{QCoh}(Z), \mathrm{QCoh}(X \times_Y \mathrm{Spec}(B))) \\ &\simeq \mathrm{Fun}^L_{\mathrm{QCoh}(Y)}(\mathrm{QCoh}(Z), \mathrm{QCoh}(X)), \end{aligned} \tag{2.129}$$

where we used descent, and that $\mathrm{Fun}^L_{\mathrm{QCoh(Y)}}(-,-)$ commutes with limits in the second variable (it is right adjoint to the Lurie tensor product). □

In what follows we let

$$\mathcal{K}_{\mathrm{QCoh},Y,\mathrm{tr}} \subseteq \mathcal{K}_{\mathrm{QCoh},Y} \tag{2.130}$$

denote the full subcategory (in the sense of $\mathrm{Mod}_{\mathrm{QCoh}(Y)}\,\mathsf{Pr}^L_{\mathsf{st}}$-enriched categories, see [HM24, Example C.1.13]), spanned by objects whose morphism to Y is transformable.

Corollary 2.3.24. *Assume that $Y \in \mathsf{Stk}$ has affine diagonal. Then*

$$\Psi_Y|_{\mathrm{tr}} : \mathcal{K}_{\mathrm{QCoh},Y,\mathrm{tr}} \to \mathrm{Mod}_{\mathrm{QCoh}(Y)}\,\mathsf{Pr}^L_{\mathsf{st}} \tag{2.131}$$

is a fully-faithful functor of $\mathrm{Mod}_{\mathrm{QCoh}(Y)}\,\mathsf{Pr}^L_{\mathsf{st}}$-enriched categories.

Corollary 2.3.25. *Assume that $Y \in \mathsf{Stk}$ has affine diagonal and $f : Z \to Y$ is transformable. Then the Fourier-Mukai transform gives an equivalence of monoidal categories*

$$FM : \mathrm{QCoh}(Z \times_Y Z) \xrightarrow{\sim} \mathrm{Fun}^L_{\mathrm{QCoh}(Y)}(\mathrm{QCoh}(Z), \mathrm{QCoh}(Z)), \tag{2.132}$$

with the convolution monoidal structure on the left and on the composition monoidal structure on the right.

Example 2.3.26. *In §3.1.5, we apply the results of this section to the case of derived rigid spaces, c.f. Corolllary 3.1.47.*

Chapter 3

Derived rigid geometry as relative algebraic geometry

3.1 Derived rigid geometry

Now let $K/\mathbf{Q}_p$ be a complete field extension. In this section we attempt to summarize the work of [BBKK24], and take some shortcuts, to obtain a theory of derived rigid spaces which is good enough for our purposes.

3.1.1 Derived affinoid spaces

Definition 3.1.1. *We define* $\mathsf{dAfndAlg}$ *to be the full subcategory of monoids A in $D_{\geqslant 0}(\mathsf{CBorn}_K)$ with the following properties.*

(i) $\pi_0(A)$ is a K-affinoid algebra. That is, it is the quotient of a (classical) Tate algebra in finitely many variables.

(ii) For every $m \geqslant 0$, $\pi_m(A)$ is finitely-generated as a $\pi_0(A)$-module.

Lemma 3.1.2. *(i)* $\mathsf{dAfndAlg}$ *is stable under pushouts in* $\mathsf{Alg}(D_{\geqslant 0}(\mathsf{CBorn}_K))$.

(ii) $\mathsf{dAfndAlg}$ *is stable under finite products in* $\mathsf{CAlg}(D_{\geqslant 0}(\mathsf{CBorn}_K))$.

(iii) $\mathsf{dAfndAlg}$ *is stable under retracts in* $\mathsf{CAlg}(D_{\geqslant 0}(\mathsf{CBorn}_K))$.

Proof. (i): Let $B \to A$ and $B \to C$ be morphisms in $\mathsf{dAfndAlg}$. Since π_0 is left adjoint to the inclusion of discrete objects in $\mathsf{CAlg}(D_{\geqslant 0}(\mathsf{CBorn}_K))$, it commutes with pushouts. In particular one has $\pi_0(A\widehat{\otimes}^{\mathbf{L}}_B C) \simeq \pi_0(A)\widehat{\otimes}_{\pi_0(B)}\pi_0(C)$. Now, the homology functors are valued in the *left heart* and so, the claim that $\pi_0(A\widehat{\otimes}^{\mathbf{L}}_B C)$ is an affinoid algebra will follow if we can show that the tensor product $\pi_0(A)\widehat{\otimes}^{LH}_{\pi_0(B)}\pi_0(C)$, which is taken with respect to the monoidal structure on the left heart, coincides with the completed tensor product of Banach spaces[1]. Let us set $A' := \pi_0(A), B' := \pi_0(B)$ and $C' := \pi_0(C)$. We first treat the case when $B' = K$. Let us temporarily write $T_n := K\langle x_1, \ldots, x_n\rangle$ and take a presentation

$$T_n^{\oplus l} \to T_n \twoheadrightarrow A'. \tag{3.1}$$

[1] I would like to thank Jack Kelly for explaining this argument to me.

Tensoring with C' and using flatness of the Tate algebra gives

$$T_n^{\oplus l}\widehat{\otimes}_K C' \xrightarrow{f} T_n\widehat{\otimes}_K C' \twoheadrightarrow A'\widehat{\otimes}_K^{LH} C', \tag{3.2}$$

so that $A'\widehat{\otimes}_K^{LH} C'$ is the cokernel of f. The image of f is an ideal, and all ideals in affinoid algebras are closed, so f is strict by the open mapping theorem. Hence $\operatorname{coker} f$ belongs to CBorn_K and coincides with $A'\widehat{\otimes}_K C'$ as required. In general, using this case we know that there is an exact sequence

$$A'\widehat{\otimes}_K B'\widehat{\otimes}_K C' \xrightarrow{g} A'\widehat{\otimes}_K C' \twoheadrightarrow A'\widehat{\otimes}_K^{LH} C', \tag{3.3}$$

and the image of g is once again an ideal, so g is strict by the same reasoning, and so $\operatorname{coker} g$ belongs to CBorn_K and coincides with $A'\widehat{\otimes}_{B'} C'$. Now we claim that each $\pi_m(A\widehat{\otimes}^{\mathbf{L}}_B C)$ is finitely-generated as a $\pi_0(A)\widehat{\otimes}_{\pi_0(B)}\pi_0(C)$-module. This follows from the convergence of the Tor-spectral sequence[2]

$$E^2_{pq} : \operatorname{Tor}^p_{\pi_*(B)}(\pi_*(A), \pi_*(C))_q \Rightarrow \pi_{p+q}(A\widehat{\otimes}^{\mathbf{L}}_B C), \tag{3.4}$$

(c.f. [BBKK24, Lemma 4.5.55]) combined with Noetherianity of affinoid algebras. (ii): This is clear from the isomorphism $\pi_*(\prod_{i=1}^n A_i) \cong \prod_{i=1}^n \pi_*(A_i)$ which holds for any finite collection of objects $\{A_i\}_{i=1}^n$ of objects of $\mathsf{dAfndAlg}$. (iii): If A is a retract of $B \in \mathsf{dAfnd}$ then $\pi_*(A)$ is a retract of $\pi_*(B)$, and hence $A \in \mathsf{dAfnd}$. □

Definition 3.1.3. *A morphism $A \to B$ in $\mathsf{CAlg}(D(\mathsf{CBorn}_K))$ is called a* homotopy epimorphism *if the codiagonal morphism $B\widehat{\otimes}^{\mathbf{L}}_A B \to B$ is an equivalence.*

Definition 3.1.4. *A morphism $A \to B$ in $\mathsf{CAlg}(D_{\geqslant 0}(\mathsf{CBorn}_K))$ is called* derived strong *if for every $m \geqslant 0$, the natural morphism*

$$\pi_m(A)\widehat{\otimes}^{\mathbf{L}}_{\pi_0(A)}\pi_0(B) \to \pi_m(B) \tag{3.5}$$

is an equivalence.

Definition 3.1.5. *A morphism $A \to B$ in $\mathsf{dAfndAlg}$ is a* derived rational localization *if:*

(i) $\pi_0(A) \to \pi_0(B)$ is a rational localization (of affinoid algebras in the classical sense).

(ii) $A \to B$ is derived strong.

We denote the class of derived rational localizations in $\mathsf{dAfndAlg}$ by $\mathscr{L}$.

Remark 3.1.6. *Because each $\pi_m(A)$ is finitely-generated as a $\pi_0(A)$-module, and finitely-generated $\pi_0(A)$-modules are transverse to rational localizations of $\pi_0(A)$, condition (ii) in Definition 3.1.5 is equivalent to:*

(ii)′ For every $m \geqslant 0$, the natural morphism

$$\pi_m(A)\widehat{\otimes}_{\pi_0(A)}\pi_0(B) \to \pi_m(B) \tag{3.6}$$

is an equivalence.

In the terminology of [BBKK24], one says that $A \to B$ is strong.

[2](Here and elsewhere, Tor denotes the homotopy groups of the derived *completed* tensor product).

Lemma 3.1.7. *(i) The class $\mathscr{L}$ contains all equivalences and is stable under composition.*

(ii) $\mathscr{L}$ is stable under base-change by arbitrary morphisms of $\mathsf{dAfndAlg}$.

(iii) Every morphism of $\mathscr{L}$ is a homotopy epimorphism.

Proof. (i): It is clear that $\mathscr{L}$ contains all equivalences. Let $A \to B$ and $B \to C$ be morphisms of $\mathscr{L}$. Since rational localizations are stable under composition, the composite $\pi_0(A) \to \pi_0(B) \to \pi_0(C)$ is a rational localization. It is easy to show that $A \to C$ is (derived) strong, by using that $A \to B$ and $B \to C$ are (derived) strong together with associativity of $\widehat{\otimes}$.

(iii): This is [BBKK24, Proposition 2.6.165(2)]. We first observe that, by [BBK17, Theorem 5.16], the morphism $\pi_0(A) \to \pi_0(B)$ is a homotopy epimorphism, meaning that $\pi_0(B)\widehat{\otimes}^{\mathbf{L}}_{\pi_0(A)}\pi_0(B) \xrightarrow{\sim} \pi_0(B)$ is an isomorphism. Using this, and the derived strong property (3.5), one has

$$\begin{aligned} \pi_*(B)\widehat{\otimes}^{\mathbf{L}}_{\pi_*(A)}\pi_*(B) &\simeq \pi_*(A)\widehat{\otimes}^{\mathbf{L}}_{\pi_0(A)}\pi_0(B)\widehat{\otimes}^{\mathbf{L}}_{\pi_0(A)}\pi_0(B) \\ &\simeq \pi_*(A)\widehat{\otimes}^{\mathbf{L}}_{\pi_0(A)}\pi_0(B) \\ &\simeq \pi_*(B). \end{aligned} \tag{3.7}$$

Now we consider the Tor-spectral sequence

$$E^2_{pq} : \mathrm{Tor}^p_{\pi_*(A)}(\pi_*(B), \pi_*(B))_q \Rightarrow \pi_{p+q}(B\widehat{\otimes}^{\mathbf{L}}_A B), \tag{3.8}$$

and observe that, because of (3.7), this collapses on the first page. In combination with (3.7) this gives $\pi_*(B\widehat{\otimes}^{\mathbf{L}}_A B) \cong \pi_*(B)$ which shows that $B\widehat{\otimes}^{\mathbf{L}}_A B \to B$ is an equivalence.

(ii): Let $A \to A'$ be a further morphism and let B' be defined as the pushout (using Lemma 3.1.2) in $\mathsf{dAfndAlg}$:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & \ulcorner & \big\downarrow \\ A' & \longrightarrow & B' \end{array} \tag{3.9}$$

Since π_0 commutes with pushouts and rational localizations of classical affinoid algebras are stable under base-change, we see that $\pi_0(A') \to \pi_0(B')$ is a rational localization.

Since $\pi_0(A') \to \pi_0(B')$ is a rational localization, and $\pi_m(A')$ is finitely-generated as a module over $\pi_0(A')$, the canonical morphism

$$\pi_m(A')\widehat{\otimes}^{\mathbf{L}}_{\pi_0(A')}\pi_0(B') \xrightarrow{\sim} \pi_m(A')\widehat{\otimes}_{\pi_0(A')}\pi_0(B') \tag{3.10}$$

is an equivalence. In the terminology of [BBKK24] one says that $\pi_m(A')$ *is transversal to* $\pi_0(B')$ *over* $\pi_0(A')$. Additionally, we claim that the natural morphism

$$B'\widehat{\otimes}^{\mathbf{L}}_{A'}B' \xrightarrow{\sim} B' \tag{3.11}$$

is an equivalence. Indeed, we know that $B' \simeq B\widehat{\otimes}^{\mathbf{L}}_A A'$ and by (iii) we have $B\widehat{\otimes}^{\mathbf{L}}_A B \xrightarrow{\sim} B$. Therefore, by the associativity properties of $\widehat{\otimes}^{\mathbf{L}}$, we obtain (3.11).

In order to prove (ii) what we needed to verify was that the morphism $A' \to B'$ satisfies the derived strong property (3.5). We claim that this follows from the properties (3.10) and (3.11)[3].

By basic properties of the cotangent complex, c.f. [BBKK24, Proposition 2.1.40, Proposition 2.1.41, Lemma 2.1.42], the homotopy epimorphism property (3.11) implies that the cotangent complex $\mathbf{L}_{B'/A'} \simeq 0$. Let C be defined by the pushout in $\mathsf{dAfndAlg}$ (again using Lemma 3.1.2):

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ \pi_0(A') & \longrightarrow & C \end{array} \tag{3.12}$$

We claim that the natural morphism $C \to \pi_0(B')$ induced by the universal property of pushouts, is an equivalence. It is clear, by applying π_0 to (3.12), that this is an isomorphism on π_0. Therefore it suffices to show that $\mathbf{L}_{\pi_0(B')/C} \simeq 0$. There is a fiber sequence

$$\mathbf{L}_{C/\pi_0(A')}\widehat{\otimes}^{\mathbf{L}}_C \pi_0(B') \to \mathbf{L}_{\pi_0(B')/\pi_0(A')} \to \mathbf{L}_{\pi_0(B')/C}. \tag{3.13}$$

Now $\mathbf{L}_{\pi_0(B')/\pi_0(A')} \simeq 0$ because $\pi_0(A') \to \pi_0(B')$ is a homotopy epimorphism (being a rational localization). And $\mathbf{L}_{C/\pi_0(A')} \simeq 0$ because $\pi_0(A') \to C$ is a homotopy epimorphsim, by base-change. So indeed $\mathbf{L}_{\pi_0(B')/C} \simeq 0$.

Now we know that the morphism

$$\pi_0(A')\widehat{\otimes}^{\mathbf{L}}_A B' \xrightarrow{\sim} \pi_0(B') \tag{3.14}$$

is an equivalence. Using this and the transversality property (3.10), an inductive argument as described in [BBKK24, Proposition 2.3.90], guarantees that $A' \to B'$ is derived strong. □

Corollary 3.1.8. *[Man22, Remark 2.4.4] Let f, g be composable morphisms in* $\mathsf{dAfndAlg}$*. If gf and f both belong to $\mathscr{L}$ then so does g.*

Proof. Let us write $f : A \to B$ and $g : B \to C$. One can write g as the composite

$$B \to C\widehat{\otimes}^{\mathbf{L}}_A B \simeq C\widehat{\otimes}^{\mathbf{L}}_B B\widehat{\otimes}^{\mathbf{L}}_A B \simeq C, \tag{3.15}$$

where we used that f is a homotopy epimorphism, c.f. Lemma 3.1.7(iii). The first map in (3.15) is the base change of $gf : A \to C$, which belongs to $\mathscr{L}$ by Lemma 3.1.7(ii). □

Let $K[T_1, \ldots, T_n]$ denote the polynomial algebra in n variables endowed with the *fine bornology*. As a bornological vector space one views $K[T_1, \ldots, T_n]$ as the colimit of its finite-dimensional K-subspaces each endowed with their canonical bornology. This is a projective object in $\mathsf{CAlg}(D_{\geqslant 0}(\mathsf{CBorn}_K))$. In particular, given any $B \in \mathsf{dAfndAlg}$ and any morphism $K[T_1, \ldots, T_n] \to \pi_0 B$ there exists a lift $K[T_1, \ldots, T_n] \to \pi_0 B$ making the following diagram commute up to homotopy:

$$\begin{array}{ccc} & & B \\ & \nearrow & \downarrow \\ K[T_1, \ldots, T_n] & \longrightarrow & \pi_0 B \end{array} \tag{3.16}$$

[3] Indeed, this is [BBKK24, Proposition 2.6.160]. However, in this specific setting the proof of *loc. cit.* simplifies and so we record it here.

Now suppose we are given $A \in \mathsf{dAfndAlg}$ and elements $a_0, \dots, a_n \in \pi_0 A$ such that the ideal generated by $a_0, \dots, a_n$ is all of $\pi_0 A$. This determines a morphism

$$K[T_1, \dots, T_n] \to \pi_0 A \widehat{\otimes}_K K\langle X_1, \dots, X_n \rangle \tag{3.17}$$

sending $T_i \mapsto a_0 X_i - a_i$, which, by the above discussion, lifts to

$$K[T_1, \dots, T_n] \to A \widehat{\otimes}^{\mathbf{L}}_K K\langle X_1, \dots, X_n \rangle. \tag{3.18}$$

We define

$$\begin{aligned} A' &:= (A \widehat{\otimes}^{\mathbf{L}}_K K\langle X_1, \dots, X_n \rangle)//^{\mathbf{L}}(a_0 X_1 - a_1, \dots, a_0 X_n - a_n) \\ &:= (A \widehat{\otimes}^{\mathbf{L}}_K K\langle X_1, \dots, X_n \rangle) \widehat{\otimes}^{\mathbf{L}}_{K[T_1, \dots, T_n]} K. \end{aligned} \tag{3.19}$$

We claim that:

Lemma 3.1.9. *(i) $A \to A'$ is a derived rational localization;*

(ii) Every derived rational localization of A arises in this way.

Proof. (i): It is certainly true that $\pi_0 A \to \pi_0 A'$ is a rational localization; what we need to show that that $A \to A'$ is derived strong. This is quite similar to the proof of Lemma 3.1.7(ii). By [BBKK24, Lemma 4.5.79], $A \to A'$ is a homotopy epimorphism. Therefore, the same argument as in Lemma 3.1.7(ii) shows that $\pi_0(A) \widehat{\otimes}^{\mathbf{L}}_A A' \to \pi_0(A')$ is an equivalence. Also, since each $\pi_m(A)$ is finitely-generated over $\pi_0(A)$, it is transversal to $\pi_0(A')$ over $\pi_0(A)$. Therefore [BBKK24, Proposition 2.3.90] guarantees that $A \to A'$ is derived strong. Finally, we note that this implies that $A' \in \mathsf{dAfndAlg}$ since $\pi_m(A') \cong \pi_m(A) \widehat{\otimes}_{\pi_0(A)} \pi_0(A')$ is then finitely-generated as a $\pi_0(A')$-module.

(ii): Let $A \to B$ be a derived rational localization. By Lemma 3.1.7(iii), $A \to B$ is a homotopy epimorphism, in particular it is formally étale in the sense of [BBKK24, Corollary 2.1.36].

Now $\pi_0 A \to \pi_0 B$ is a rational localization, so by the construction in (i) we can construct another rational localization $A \to B'$ which is of the form in (i), and is such that $[\pi_0 A \to \pi_0 B'] = [\pi_0 A \to \pi_0 B]$. Now the étale lifting property [BBKK24, Corollary 2.1.36], guarantees that the identification $\pi_0 B' = \pi_0 B$ lifts, uniquely up to contractible choice, to an equivalence $B' \simeq B$ under A. ☐

Let us isolate the following facts from the proof of Lemma 3.1.9:

Scholium 3.1.10. *Let $A \in \mathsf{dAfnd}$.*

(i) Let $A \to A'$, $A \to A''$ be derived rational localizations. Any isomorphism $\pi_0 A' \cong \pi_0 A''$ lifts (uniquely up to contractible choice) to an equivalence $A' \simeq A''$ under A,

(ii) Let $\pi_0 A \to B_0$ be a rational localization. Then there exists a (unique up to contractible choice) derived rational localization $A \to B$ reducing to $\pi_0 A \to B_0$ on π_0.

Definition 3.1.11. *(i) We define dAfnd to be the opposite ∞-category to $\mathsf{dAfndAlg}$. The objects of dAfnd are denoted by $\mathrm{dSp}(A)$, for $A \in \mathsf{dAfndAlg}$.*

(ii) We say that a morphism $\mathrm{dSp}(B) \to \mathrm{dSp}(A)$ in dAfnd is a rational subdomain *of $\mathrm{dSp}(A)$ if $[A \to B] \in \mathscr{L}$.*

Now we define some Grothendieck topologies as follows.

Definition 3.1.12. *Let $X = \mathrm{dSp}(A)$ be a derived affinoid rigid space.*

(i) The small weak analytic site *of X, is the ∞-site with:*

(a) underlying ∞-category given by the full subcategory of $\mathsf{dAfnd}_{/X}$ on rational subdomains $\mathrm{dSp}(B) \to \mathrm{dSp}(A)$.

(b) covering sieves generated by finite families of rational subdomains $\{\mathrm{dSp}(B_i) \to \mathrm{dSp}(B)\}_{i=1}^n$ such that $\{\mathrm{Sp}(\pi_0(B_i)) \to \mathrm{Sp}(\pi_0(B))\}_{i=1}^n$ is an admissible covering of the classical rigid space $\mathrm{Sp}(\pi_0(B))$ in the weak G-topology.

(ii) The big weak analytic site *on dAfnd, is the ∞-site with:*

(a) underlying ∞-category given by dAfnd;

(b) covering sieves generated given by finite families of rational subdomains $\{\mathrm{dSp}(B_i) \to \mathrm{dSp}(B)\}_{i=1}^n$ such that $\{\mathrm{Sp}(\pi_0(B_i)) \to \mathrm{Sp}(\pi_0(B))\}_{i=1}^n$ is an admissible covering of the classical rigid space $\mathrm{Sp}(\pi_0(B))$ in the weak G-topology.

Definition 3.1.13. *Let $X = \mathrm{dSp}(A)$ be a derived affinoid rigid space. We define the ∞-category of quasicoherent sheaves as*

$$\mathrm{QCoh}(\mathrm{dSp}(A)) := \mathrm{Mod}_A(D(\mathsf{CBorn}_K)), \tag{3.20}$$

where the latter is the category of modules over the monoid $A \in \mathsf{CAlg}(D(\mathsf{CBorn}_K))$.

For a morphism $f : \mathrm{dSp}(A) \to \mathrm{dSp}(B)$ in dAfnd the induced pullback functor

$$f^* : \mathrm{QCoh}(\mathrm{dSp}(B)) \to \mathrm{QCoh}(\mathrm{dSp}(A)) \tag{3.21}$$

is left adjoint to the restriction of scalars

$$f_* : \mathrm{QCoh}(\mathrm{dSp}(A)) \to \mathrm{QCoh}(\mathrm{dSp}(B)). \tag{3.22}$$

In particular via the pullbacks we obtain a functor

$$\mathrm{QCoh} : \mathsf{dAfnd}^{\mathsf{op}} \to \mathsf{CAlg}(\mathsf{Pr}^L_{\mathsf{st}}). \tag{3.23}$$

Lemma 3.1.14. *(i) The functor $\mathrm{QCoh} : \mathsf{dAfnd}^{\mathsf{op}} \to \mathsf{CAlg}(\mathsf{Pr}^L_{\mathsf{st}})$ extends to a six-functor formalism*

$$\mathrm{QCoh} : \mathrm{Corr}(\mathsf{dAfnd}, \mathrm{all})^{\otimes} \to \mathsf{Pr}^{L,\otimes}_{\mathsf{st}} \tag{3.24}$$

such that for every morphism f in dAfnd one has $f_* = f_!$.

(ii) For every morphism f in dAfnd functor f_* is conservative and colimit-preserving.

Proof. (i): The proof is identical to Proposition 2.3.13. (ii) This is clear because f_* identifies with the forgetful functor at the level of modules. □

Definition 3.1.15. *We define*

$$\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd}) := \mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd}, \infty\mathsf{Grpd}) \tag{3.25}$$

to be the ∞-category of sheaves on the ∞-site dAfnd equipped with the weak topology.

Lemma 3.1.16. *(i) For every derived affinoid rigid space $X = \mathrm{dSp}(A)$ the functor $\mathrm{dSp}(A) : \mathsf{dAfnd}^{\mathsf{op}} \to \infty\mathsf{Grpd}$ represented by $\mathrm{dSp}(A)$*[4] *is a sheaf on dAfnd in the weak topology. That is to say, this topology is subcanonical.*

(ii) The functor $\mathrm{QCoh} : \mathsf{dAfnd}^{\mathsf{op}} \to \mathsf{CAlg}(\mathsf{Pr}^L_{\mathsf{st}})$ is a sheaf on dAfnd in the weak topology.

Proof. We prove (ii) first. Let $\{\mathrm{dSp}(B_i) \to \mathrm{dSp}(B)\}_{i=1}^n$ be a covering in the weak topology. Let $\mathcal{I}$ be the collection of finite subsets $I \subseteq \{1, \dots, n\}$ (we always view such I as being totally ordered). By definition, $\{\mathrm{Sp}(\pi_0(B_i)) \to \mathrm{Sp}(\pi_0(B))\}_{i=1}^n$ is a covering in the classical weak topology. The acyclicity of the ordered Čech complex together with the Dold-Kan correspondence [Lur17, Example 1.2.4.10] implies that

$$\begin{aligned}\pi_0(B) \xrightarrow{\sim} &\lim_{I=(i_1,\dots,i_k)\in\mathcal{I}} \pi_0(B_{i_1})\widehat{\otimes}_{\pi_0(B)}\pi_0(B_{i_2})\dots\widehat{\otimes}_{\pi_0(B)}\pi_0(B_{i_k}) \\ \simeq &\lim_{I=(i_1,\dots,i_k)\in\mathcal{I}} \pi_0(B_{i_1}\widehat{\otimes}^{\mathbf{L}}_B B_{i_2}\dots\widehat{\otimes}^{\mathbf{L}}_B B_{i_k})\end{aligned} \tag{3.26}$$

is an equivalence in the ∞-category $\mathrm{QCoh}(\mathrm{dSp}(\pi_0(A)))$. Using that the class $\mathscr{L}$ of derived rational localizations is stable under base change, c.f. Lemma 3.1.7(ii), we know that each $B \to B_{i_1}\widehat{\otimes}^{\mathbf{L}}_B B_{i_1}\dots\widehat{\otimes}^{\mathbf{L}}_B B_{i_k}$ satisfies the derived strong property (3.5). Therefore if we apply the functor $\pi_q(B)\widehat{\otimes}^{\mathbf{L}}_{\pi_0(B)} -$ to both sides of (3.26), using that the limit is finite and the categories are stable, we see that

$$\pi_q(B) \xrightarrow{\sim} \lim_{I=(i_1,\dots,i_k)\in\mathcal{I}} \pi_q(B_{i_1}\widehat{\otimes}^{\mathbf{L}}_B B_{i_2}\dots\widehat{\otimes}^{\mathbf{L}}_B B_{i_k}) \tag{3.27}$$

is an equivalence in $\mathrm{QCoh}(\mathrm{dSp}(\pi_0(B)))$. In particular

$$H^p\Big(\lim_{(i_1,\dots,i_k)\in\mathcal{I}} \pi_q(B_{i_1}\widehat{\otimes}^{\mathbf{L}}_B B_{i_2}\dots\widehat{\otimes}^{\mathbf{L}}_B B_{i_k})\Big) \cong 0 \quad \text{for} \quad p > 0. \tag{3.28}$$

This implies that the spectral sequence

$$H^p\Big(\lim_{(i_1,\dots,i_k)\in\mathcal{I}} \pi_q(B_{i_1}\widehat{\otimes}^{\mathbf{L}}_B\dots\widehat{\otimes}^{\mathbf{L}}_B B_{i_k})\Big) \Rightarrow \pi_{q-p}\Big(\lim_{(i_1,\dots,i_k)\in\mathcal{I}} B_{i_1}\widehat{\otimes}^{\mathbf{L}}_B\dots\widehat{\otimes}^{\mathbf{L}}_B B_{i_k}\Big) \tag{3.29}$$

degenerates and gives an isomorphism

$$\pi_q\Big(\lim_{(i_1,\dots,i_k)\in\mathcal{I}} B_{i_1}\widehat{\otimes}^{\mathbf{L}}_B B_{i_1}\dots\widehat{\otimes}^{\mathbf{L}}_B B_{i_k}\Big) \cong \mathrm{eq}\Big(\prod_{i=1}^n \pi_q(B_i) \rightrightarrows \prod_{1\leqslant i<j\leqslant n} \pi_q(B_i\widehat{\otimes}^{\mathbf{L}}_B B_j)\Big). \tag{3.30}$$

On the other hand since $B \to B_{i_1}\widehat{\otimes}^{\mathbf{L}}_B B_{i_1}\dots\widehat{\otimes}^{\mathbf{L}}_B B_{i_k}$ satisfies the derived strong property (3.5) and $\pi_q(B)$ is transversal to rational localizations over $\pi_0(B)$ since it is finitely-generated, we see that

$$\begin{aligned}\pi_q(B_i\widehat{\otimes}^{\mathbf{L}}_B B_j) &\simeq \pi_q(B)\widehat{\otimes}^{\mathbf{L}}_{\pi_0(B)}\pi_0(B_i\widehat{\otimes}^{\mathbf{L}}_B B_j) \\ &\simeq \pi_q(B)\widehat{\otimes}_{\pi_0(B)}\pi_0(B_i)\widehat{\otimes}_{\pi_0(B)}\pi_0(B_j).\end{aligned} \tag{3.31}$$

[4] Via the ∞-categorical Yoneda embedding.

Using (3.31) in the right-side of (3.30) we obtain

$$\begin{aligned} &\operatorname{eq}\Big(\prod_{i=1}^{n}\pi_q(B_i) \rightrightarrows \prod_{1\leqslant i<j\leqslant n}\pi_q(B_i\widehat{\otimes}^{\mathbf{L}}_B B_j)\Big)\\ &\cong \operatorname{eq}\Big(\prod_{i=1}^{n}\pi_q(B)\widehat{\otimes}_{\pi_0(B)}\pi_0(B_i) \rightrightarrows \prod_{1\leqslant i<j\leqslant n}\pi_q(B)\widehat{\otimes}_{\pi_0(B)}\pi_0(B_i)\widehat{\otimes}_{\pi_0(B)}\pi_0(B_j)\Big)\\ &\cong \pi_q(B), \end{aligned} \tag{3.32}$$

where in the last line we used the classical theorem of descent for coherent sheaves on affinoid rigid spaces. Putting this all together we deduce that for each $q \geqslant 0$, the morphism

$$\pi_q(B) \to \pi_q\Big(\lim_{(i_1,\ldots,i_k)\in\mathcal{I}} B_{i_1}\widehat{\otimes}^{\mathbf{L}}_B \ldots \widehat{\otimes}^{\mathbf{L}}_B B_{i_k}\Big) \tag{3.33}$$

is an isomorphism and therefore the natural morphism

$$B \xrightarrow{\sim} \lim_{(i_1,\ldots,i_k)\in\mathcal{I}} B_{i_1}\widehat{\otimes}^{\mathbf{L}}_B \ldots \widehat{\otimes}^{\mathbf{L}}_B B_{i_k}, \tag{3.34}$$

is an equivalence. This shows that the canonical morphism $B \to \prod_{i=1}^n B_i$ is descendable in the sense of Mathew [Mat16, §3.3]. The result of *loc. cit.* then implies that if one sets $Y := \coprod_{i=1}^n \mathrm{dSp}(B_i) \to \mathrm{dSp}(B) =: X$ then the natural morphism

$$\mathrm{QCoh}(X) \xrightarrow{\sim} \lim_{[m]\in\Delta} \mathrm{QCoh}(Y^{m+1/X}) \tag{3.35}$$

is an equivalence, proving (ii). Looking at the unit object we see that

$$B \xrightarrow{\sim} \lim_{[m]\in\Delta}\big(\prod_{i=1}^{n} B_i\big)^{\widehat{\otimes}^{\mathbf{L}}_B (m+1)} \tag{3.36}$$

is an equivalence. If $A \in \mathsf{dAfndAlg}$ is a derived affinoid algebra, we can apply the functor $\mathrm{Map}_{\mathsf{dAfndAlg}}(A, -)$ to both sides of (3.36) to deduce (i). □

Remark 3.1.17 (Notational remark)**.** *In derived geometry, it appears to be conventional to mix homological and cohomological indexing conventions, that is π_i stands for the* homology *functors, H^i stands for the* cohomology *functors, and they are related by $\pi_i = H^{-i}$ for $i \in \mathbf{Z}$. Usually, one uses π_i for the homotopy groups of commutative algebra objects, and H^i for the cohomology groups of "linear" objects like cochain complexes (though this is not a hard and fast rule). The historical reason for this is that "derived rings" are often modelled on simplicial commutative rings, which have "homotopy groups", whereas algebraic geometers prefer cohomological indexing convention for derived categories of quasi-coherent sheaves.*

Remark 3.1.18. *Let $X = \mathrm{dSp}(A)$ be a derived affinoid rigid space and let $\{U_i \to X\}_{i=1}^n$ be a covering in the weak topology. Let $\mathcal{I}$ be the family of finite nonempty subsets $I \subseteq \{1,\ldots,n\}$ ordered by inclusion. For $I = (i_0,\ldots,i_k) \in \mathcal{I}$, we set $U_I := \bigcap_{j\in I} U_j := U_{i_0} \times_X \cdots \times_X U_{i_k}$. In the course of proving Lemma 3.1.16 we obtained two facts.*

(i) There is a canonical equivalence

$$\operatorname*{colim}_{I\in\mathcal{I}} U_I \xrightarrow{\sim} X \tag{3.37}$$

in $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})$. This is a consequence of (3.34).

(ii) Set $Y := \coprod_{i=1}^n U_i$*. Then there is a canonical equivalence*

$$\operatorname*{colim}_{[m]\in\Delta^{\mathrm{op}}} Y^{m+1/X} \xrightarrow{\sim} X \tag{3.38}$$

in $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})$*. This is a consequence of* (3.36)*.*

3.1.2 The gluing procedure

Now in a similar manner to [Man22, §2.4] we will glue derived affinoid rigid spaces to obtain our desired category of derived rigid spaces. Since the weak topology is subcanonical we can view $\mathsf{dAfnd} \subseteq \mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})$ as a full subcategory. In fact we will slightly abuse the terminology in order to make the following definition.

Definition 3.1.19. *(i) A* affinoid derived rigid space *is an object of* $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})$ *which is isomorphic to* $\mathrm{dSp}(A)$ *for some* $A \in \mathsf{dAfndAlg}$*.*

(ii) Let $X = \mathrm{dSp}(A)$ *be an affinoid derived rigid space. An analytic subspace* $U \hookrightarrow X$ *is a subsheaf* U *of* X *such that:*

- ⋆ *There exists a small collection* $\{\mathrm{dSp}(A_i)\}_{i\in\mathcal{I}}$ *of derived affinoid spaces and an effective epimorphism* $\coprod_{i\in\mathcal{I}} \mathrm{dSp}(A_i) \to U$ *in* $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})$ *such that each* $\mathrm{dSp}(A_i) \to \mathrm{dSp}(A)$ *is a rational subdomain.*

(iii) Let $X \in \mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})$*. An analytic subspace* $Y \hookrightarrow X$ *is a subsheaf such that for every affinoid derived rigid space* $\mathrm{dSp}(A) \to X$ *mapping to* X*, the pullback* $Y \times_X \mathrm{dSp}(A) \hookrightarrow \mathrm{dSp}(A)$ *is an analytic subspace in the sense of (ii). The morphism* $Y \to X$ *is then called an* open immersion*.*

(iv) A derived rigid space *is an object* $X \in \mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})$ *such that there exists a small collection* $\{\mathrm{dSp}(A_i)\}_{i\in\mathcal{I}}$ *of derived affinoid subspaces such that* $\coprod_{i\in\mathcal{I}} \mathrm{dSp}(A_i) \to X$ *is an effective epimorphism. We denote the full subcategory of* $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})$ *on derived rigid spaces by* dRig*.*

We remark that the category dRig admits all fiber products. The terminal object is $\mathrm{dSp}(K)$. We can formally define some Grothendieck topologies as follows.

Definition 3.1.20. *Let* $X \in \mathsf{dRig}$ *be a derived rigid space.*

(i) The small strong analytic site *of* X*, is the* ∞*-site with:*

- *(a) underlying* ∞*-category given by the full subcategory of* $\mathsf{dRig}_{/X}$ *on analytic subspaces* $Y \hookrightarrow X$*;*
- *(b) covering sieves generated by small families of analytic subspaces* $\{Y_i \hookrightarrow X\}_{i\in\mathcal{I}}$ *such that* $\coprod_{i\in\mathcal{I}} Y_i \to X$ *is an effective epimorphism.*

(ii) The big strong analytic site *on* dRig*, is the* ∞*-site with:*

- *(a) underlying* ∞*-category given by* dRig*;*
- *(b) covering sieves generated by small families of analytic subspaces* $\{Y_i \hookrightarrow X\}_{i\in\mathcal{I}}$ *such that* $\coprod_{i\in\mathcal{I}} Y_i \to X$ *is an effective epimorphism.*

3.1.3 The underlying topological space

In order to make this theory more workable we will associate to each $X \in \mathsf{dRig}$ a topological space. We always assume the axiom of choice[5].

[5]This is necessary in order to know that every filter is contained in a maximal filter.

Definition 3.1.21. *An object $X \in \mathsf{dRig}$ is called* classical *if it admits a covering $\coprod_{i\in\mathcal{I}} \mathrm{dSp}(A_i) \to X$ by affinoid subspaces such that all the A_i are discrete.*

Lemma 3.1.22. *The inclusion of classical rigid spaces admits a right adjoint $X \mapsto X_0$ which extends $\mathrm{dSp}(A) \mapsto \mathrm{dSp}(\pi_0(A))$. $Y \hookrightarrow X$ is an analytic subspace if and only if $Y_0 \hookrightarrow X_0$ is an analytic subspace.*

Proof. Let us temporarily denote $\mathscr{X} := \mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})$. The functor $(-)_0 : \mathsf{dAfnd} \to \mathsf{dAfnd} : \mathrm{dSp}(A) \mapsto \mathrm{dSp}(\pi_0 A)$ preserves fiber products. For every finite covering family $\{\mathrm{dSp}(B_i) \to \mathrm{dSp}(B)\}_{i=1}^n$ in the weak topology, by Definition 3.1.12 one has that $\{\mathrm{dSp}(\pi_0 B_i) \to \mathrm{dSp}(\pi_0 B)\}_{i=1}^n$ is a covering family in the weak topology. Since the topology is subcanonical by Lemma 3.1.16, this implies that $\coprod_{i=1}^n \mathrm{dSp}(\pi_0 B_i) \to \mathrm{dSp}(\pi_0 B)$ is an effective epimorphism in $\mathscr{X}$. Therefore one may use the local Yoneda embedding [Lur09b, Proposition 6.2.30] to extend $(-)_0$ to a colimit-preserving left-exact functor $(-)_0 : \mathscr{X} \to \mathscr{X}$. In particular $(-)_0$ preserves effective epimorphisms and subobjects, and therefore sends objects of $\mathsf{dRig} \subseteq \mathscr{X}$ to classical rigid spaces (in the sense of the above definition), and also restricts to the identity on classical rigid spaces. Let Y be a classical rigid space and let $X \in \mathsf{dRig}$. Then $(-)_0$ induces $\mathrm{Map}(Y, X) \to \mathrm{Map}(Y, X_0)$. We claim that this is an equivalence. Fix a covering $\mathcal{U} = \{U_i\}$ of X by derived affinoids. Given a covering $\mathcal{V} = \{V_j\}$ of Y by classical affinoids one can consider the full subspace $\mathrm{Map}^{\mathcal{V}\mathcal{U}}(Y, X) \subseteq \mathrm{Map}(Y, X)$ spanned by morphisms f such that $\mathcal{V}$ refines the pullback of $\mathcal{U}$ along f. One then has $\mathrm{colim}_{\mathcal{V}} \mathrm{Map}^{\mathcal{V}\mathcal{U}}(Y, X) \simeq \mathrm{Map}(Y, X)$, where the system of coverings $\mathcal{V}$ is ordered by refinement. Hence one can reduce the claim to the case when X is a derived affinoid and then further to when Y is a classical affinoid. The claim then follows since π_0 is a left adjoint at the level of algebra. □

Now in a similar manner to [CS19b, Lecture XIV] we will associate a topological space to every $X \in \mathsf{dRig}$ as follows. Viewed as an object of $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})$, the collection $\mathrm{Sub}(X)$ of subobjects of X forms a locale, as can be seen by applying the results of [Lur09b, §6.4.5] to the ∞-topos $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})_{/X}$. We will write the operations of join and meet in this locale as $\cup$ and $\cap$, respectively. Now we have the following Lemma.

Lemma 3.1.23. *With respect to the same operations of join and meet, the poset of analytic subspaces of X, in the sense of Definition 3.1.12, also forms a locale.*

Proof. It is clear that the meet of two analytic subspaces of X is again an analytic subspace: one has $U \cap V = U \times_X V$. The join of two objects $U, V \in \mathrm{Sub}(X)$ is given explicitly as

$$U \cup V = U \coprod_{U\cap V} V, \tag{3.39}$$

from which one can of course deduce the formula for finite joins. More generally, if $\{U_i\}_{i\in\mathcal{I}}$ is a small family of analytic subspaces of X, then one has explicitly

$$\bigcup_{i\in\mathcal{I}} U_i = \operatorname*{colim}_I \bigcup_{i\in I} U_i, \tag{3.40}$$

where the colimit runs over the finite subsets $I \subseteq \mathcal{I}$. Now we recall that colimits in any ∞-topos are universal (this is one of the Giraud–Rezk–Lurie axioms [Lur09b, Theorem 6.1.0.6]), so that if $Y = \mathrm{dSp}(A) \to X$ is a derived affinoid mapping to X, by (3.39) and (3.40) one has

$$\Big(\bigcup_{i\in\mathcal{I}} U_i\Big) \times_X Y = \bigcup_{i\in\mathcal{I}} U_i', \tag{3.41}$$

where $U_i' := U_i \times_X Y$. By assumption each U_i' is an analytic subspace of the affinoid space Y, in the sense of Definition 3.1.19(ii). By looking at the formulas (3.39) and (3.40) with U_i' in place of U_i, it is clear that $\bigcup_{i\in\mathcal{I}} U_i'$ is an analytic subspace of the affinoid space Y, in the sense of Definition 3.1.19(ii). That is, if $\{V_{ij}'\}_{j\in\mathcal{J}(i)}$ is a small collection of rational opens of Y such that $\coprod_{j\in\mathcal{J}(i)} V_{ij}' \to U_i'$ is an effective epimorphism, then

$$\coprod_{i\in\mathcal{I}} \coprod_{j\in\mathcal{J}(i)} V_{ij}' \to \bigcup_{i\in\mathcal{I}} U_i' \tag{3.42}$$

is an effective epimorphism. This completes the proof, c.f. Definition 3.1.19(iii). □

We denote the locale of analytic subspaces of X by $\mathrm{An}(X)$. It is an immediate consequence of Definition 3.1.20(i) that for any ∞-category $\mathscr{D}$ admitting small limits there is a canonical equivalence of ∞-categories

$$\mathrm{Shv}_{\mathrm{strong}}(X, \mathscr{D}) \simeq \mathrm{Shv}(\mathrm{An}(X), \mathscr{D}), \tag{3.43}$$

where the latter is the category of sheaves on the locale. A basis for the locale $\mathrm{An}(X)$ is given by the affinoid analytic subspaces of X, which are quasi-compact. In particular, the locale $\mathrm{An}(X)$ is locally compact. Therefore by Hoffman-Lawson duality [Joh86, VII,§4] the locale is spatial; that is, if one sets

$$|X| := \mathrm{pt}(\mathrm{An}(X)) \tag{3.44}$$

to be the topological space of *points*, i.e., the completely prime filters on the locale, then there is a canonical isomorphism of locales $\Omega(|X|) \cong \mathrm{An}(X)$. Here Ω is the functor which sends a topological space to its locale of open subsets. In particular, for any ∞-category $\mathscr{D}$ admitting small limits one obtains a canonical equivalence of ∞-categories

$$\mathrm{Shv}(\mathrm{An}(X), \mathscr{D}) \simeq \mathrm{Shv}(|X|, \mathscr{D}). \tag{3.45}$$

It follows by functoriality of the above constructions, that $|\cdot|$ determines a covariant functor $|\cdot| : \mathsf{dRig} \to \mathsf{Top}$, where the latter is (the nerve of) the ordinary category of topological spaces. As it arises from a spatial locale, $|\cdot|$ factors through sober topological spaces. We will prove that this topological space obeys one of the principles of derived geometry, which says that "X_0 contains all the geometry". To prove this we will essentially follow the same recipe as [Man22, §2.9].

Lemma 3.1.24. *(i) Let $\{\mathrm{dSp}(A_i) \to \mathrm{dSp}(A)\}_{i=1}^n$ be a finite collection of derived rational subdomains of $\mathrm{dSp}(A)$. Then $\coprod_{i=1}^n |\mathrm{dSp}(A_i)| \to |\mathrm{dSp}(A)|$ is a surjective morphism of topological spaces if and only if $\coprod_{i=1}^n |\mathrm{dSp}(\pi_0 A_i)| \to |\mathrm{dSp}(\pi_0 A)|$ is a surjective morphism of topological spaces.*

(ii) Let $X \in \mathsf{dRig}$ and let $U, U' \subseteq X$ be analytic subspaces. Then one has $|U'| \subseteq |U|$ if and only if $|(U')_0| \subseteq |U_0|$. In particular $|U| = |U'|$ if and only if $|(U')_0| = |U_0|$.

Proof. (i): This is an immediate consequence of Definition 3.1.12. (ii): This follows from (i). □

Lemma 3.1.25. *Let $X = \mathrm{dSp}(A)$ be a derived affinoid space and let $V \subseteq X_0$ be a rational subdomain. Then there exists a (unique up to contractible choice) rational subdomain $U \subseteq X$ such that $U_0 = V$ as subobjects of X_0.*

Proof. This is Scholium 3.1.10(ii). ☐

Corollary 3.1.26. *Let $X \in \mathsf{dRig}$ and let $V \subseteq |X_0|$ be an open subset. Then there exists an analytic subspace $U \subseteq X$ such that $|U_0| = V$.*

Proof. The problem is local on X, so we may assume that X is a derived affinoid. Then the claim follows from Lemma 3.1.25, since the rational subdomains form a basis for the topology. ☐

Theorem 3.1.27. *Let $X \in \mathsf{dRig}$. The functor $(-)_0$ induces an isomorphism of topological spaces $|X_0| \xrightarrow{\sim} |X|$.*

Proof. This follows from Lemma 3.1.24 and Lemma 3.1.25, using that $|\cdot|$ takes values in sober topological spaces (so that $|X|, |X_0|$ are determined by their lattice of opens), and using also that the open subsets biject with analytic subspaces. ☐

Remark 3.1.28. *(i) Let $X = \mathrm{dSp}(A)$ be an affinoid derived rigid space. The lattice* $\mathrm{Special}(X)$ *of finite unions of rational subdomains forms a basis for the basis for the locale* $\mathrm{An}(X)$*, and so completely prime filters on* $\mathrm{An}(X)$ *biject with prime filters on the lattice* $\mathrm{Special}(X)$*. We note that* $\mathrm{Special}(X)$ *is a bounded distributive lattice and hence by the Stone representation theorem for distributive lattices, c.f. [Joh86, II, §3], we see that $|X|$ is a spectral topological space.*[6] *We may also make the following definition. Let $|X|_{Ber}$ be the collection of maximal filters on* $\mathrm{Special}(X)$*. This is naturally a compact topological space [Joh86, II, §3] equipped with an injective morphism $|X|_{Ber} \to |X|$. This construction defines a functor $|\cdot|_{Ber} : \mathsf{dAfnd} \to \mathsf{Top}$, equipped with a natural transformation $|\cdot|_{Ber} \to |\cdot|$.*

(ii) In the situation of (ii), [Joh86, II, Proposition 3.7] asserts that the following are equivalent for a derived affinoid space X:

- *(a) Whenever $U_1, U_2 \in$* $\mathrm{Special}(X)$ *with $U_1 \cap U_2 = \emptyset$, we can find $V_1, V_2 \in$* $\mathrm{Special}(X)$ *with $V_1 \cup V_2 = X$, $V_1 \cap U_2 = \emptyset$ and $U_1 \cap V_2 = \emptyset$, c.f.*[7] *[Joh86, II, †3.6].*
- *(b) $|X|_{Ber}$ is Hausdorff.*
- *(c) Every prime filter*[8] *$x \in |X|$ is contained in a unique maximal filter $r(x) \in |X|_{Ber}$.*
- *(d) The canonical morphism $|X|_{Ber} \to |X|$ is a split monomorphism.*

In (d) the retraction can be chosen as the canonical map $x \mapsto r(x)$ coming from (c). If any of the above equivalent conditions are satisfied then (it is trivial to see that) the topology on $|X|_{Ber}$ is the quotient topology induced by the surjection $r : |X| \twoheadrightarrow |X|_{Ber}$.

(iii) In the situation of (ii), suppose further that A is discrete, so that X is a classical affinoid. Then the results of [vdPS95] and [Hub93, Corollary 4.5] imply that there are canonical isomorphisms

$$|\operatorname{Spa}(A, A^\circ)| \cong |X| \quad \text{and} \quad \mathcal{M}(X) \cong |X|_{Ber} \tag{3.46}$$

[6]For arbitrary, non-affinoid X this shows that $|X|$ is locally spectral.
[7]Note that our situation is dual to Johnstone's, because we use filters rather than *ideals* of lattices.
[8]Here we have identified points of $|X|$ with prime filters on $\mathrm{Special}(X)$.

of functors on the category of classical affinoid algebras, where $|\operatorname{Spa}(A, A^{\circ})|$ *is the Huber spectrum and* $\mathcal{M}(X)$ *is the Berkovich spectrum. These isomorphisms fit into the commutative diagram*

$$\begin{array}{ccc} |\operatorname{Spa}(A, A^{\circ})| & \xrightarrow{\cong} & |X| \\ \uparrow & & \uparrow \\ \mathcal{M}(A) & \xrightarrow{\cong} & |X|_{Ber} \end{array} \tag{3.47}$$

of functors on the category of classical affinoid algebras. By [vdPS95, Lemma 6], the morphism $\mathcal{M}(X) \to |\operatorname{Spa}(A, A^{\circ})|$ *is a split monomorphism; therefore the equivalent conditions in (ii) above are satisfied.*

(iv) Now if $X \in \mathsf{dAfnd}$ *is a derived affinoid space then it follows from Theorem 3.1.27 and (iii) that there is a commutative diagram*

$$\begin{array}{ccc} |\operatorname{Spa}(\pi_0 A, (\pi_0 A)^{\circ})| & \xrightarrow{\cong} & |X| \\ \uparrow & & \uparrow \\ \mathcal{M}(\pi_0 A) & \xrightarrow{\cong} & |X|_{Ber} \end{array} \tag{3.48}$$

of functors on dAfnd*. In particular, we see that the equivalent conditions in (ii) are satisfied for arbitrary derived affinoid spaces* X*.*

(v) As a corollary of [Lur09b, Proposition 7.3.6.10] and Theorem 3.1.27 one sees that the canonical morphism of ∞*-topoi* $r_* : \operatorname{Shv}(|X|) \to \operatorname{Shv}(|X|_{Ber})$ *induced by* $r : |X|_{Ber} \to |X|$*, is cell-like. This can be regarded as a higher-categorical generalization of the characterisation of sheaves of sets on the Berkovich space as "overconvergent sheaves".*

(vi) The definition of $|X|_{Ber}$ *given in (i) above generalises to qcqs derived rigid spaces. It is not so clear how to generalize this to arbitrary derived rigid spaces. The problem is that it is not necessarily true that every maximal filter on the locale* $\operatorname{An}(X)$ *is completely prime*[9] *but for qcqs derived rigid spaces one can circumvent this problem by using that the lattice has a canonical finitary basis formed by quasi-compact subspaces.*

3.1.4 Quasi-compact and quasi-separated morphisms

Definition 3.1.29. *(i) An object* $X \in \mathsf{dRig}$ *is called* quasi-compact *if every cover of* X *in the strong topology admits a finite subcover.*

(ii) A morphism $f : X \to Y$ *in* dRig *is called* quasi-compact *if, for every quasi compact* Z *with a morphism* $Z \to Y$*, the pullback* $X \times_Y Z$ *is quasi-compact.*

(iii) A morphism $f : X \to Y$ *in* dRig *is called* quasi-separated *if its diagonal* $\Delta_f : X \to X \times_Y X$ *is quasi-compact.*

(iv) A object $X \in \mathsf{dRig}$ *is called* quasi-separated *if the structure morphism* $X \to \operatorname{dSp}(K)$ *is quasi-separated.*

[9] In [vdPS95, §5], the authors do not seem to consider this problem.

(v) We abbreviate quasi-compact and quasi-separated *to* qcqs.

Lemma 3.1.30. *(i) The classes of quasi-compact and quasi-separated morphisms are stable under base change and composition.*

(ii) Let f and g be composable morphisms in dRig*. If fg is quasi-compact and f is quasi-separated then g is quasi-compact.*

Proof. (i): Omitted. (ii): Let us write $f : Y \to X$ and $g : Z \to Y$. We can factor g as $Z \to Z \times_X Y \to Y$. Here the first map is the base-change of Δ_f, and the second map is the base-change of fg. Therefore, since quasi-compact morphisms are stable under base-change and composition, we see that g is quasi-compact. □

Remark 3.1.31. *(i) The notions defined in Definition 3.1.29 make sense in any ∞-site. In particular they make sense in* $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})$ *equipped with the effective epimorphism topology.*

(ii) The weak topology on dAfnd *is a* finitary *Grothendieck topology [Lur11, Definition 3.17]. Therefore, by [Lur11, Proposition 3.19] we obtain the following. Every object of* dAfnd *is quasi-compact and quasi-separated, when viewed as an object of* $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})$*. In particular every affinoid derived rigid space is quasi-compact object in* dRig*. Moreover, the ∞-topos* $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})$ *is* locally coherent *[Lur11, Definition 3.12].*

Lemma 3.1.32. *Let $f : X \to Y$ be a morphism in* dRig*. The following are equivalent.*

(i) f is quasi-compact.

(ii) For every affinoid subspace $U \subseteq X$, $Y \times_X U$ is quasi-compact.

(iii) There exists a covering of $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ of Y by affinoid subspaces such that for each $i \in \mathcal{I}$, $X \times_Y U_i$ is quasi-compact.

Proof. (i) $\implies$ (ii) $\implies$ (iii) is obvious. We prove (iii) $\implies$ (i). Let $g : Z \to Y$ be a morphism from a quasi-compact object $Z \in \mathsf{dRig}$. Choose a finite covering $\mathcal{V} = \{V_j\}_{j\in\mathcal{J}}$ of Z by affinoid subspaces such that $\mathcal{V}$ refines the pullback of $\mathcal{U}$ along g. Then $\{X \times_Y V_j\}_{j\in\mathcal{J}}$ is a finite covering of $X \times_Y Z$ by quasi-compact subspaces. Therefore $X \times_Y Z$ is quasi-compact. □

Lemma 3.1.33. *Let $f : X \to Y$ be a morphism in* dRig*. The following are equivalent.*

(i) f is quasi-separated.

(ii) For every pair of affinoid subspaces $U, V \subseteq X$ mapping into a commmon affinoid subspace of Y, $U \times_X V$ is a admits a finite cover by affinoid subspaces of X.

(iii) There exists a cover $\{U_i\}_{i\in\mathcal{I}}$ of Y by affinoid subspaces and for each $i \in \mathcal{I}$ a cover $\{V_j\}_{j\in\mathcal{J}(i)}$ of $X \times_Y U_i$ by affinoid subspaces such that for every $j, j' \in \mathcal{J}(i)$, $V_j \times_X V_{j'}$ admits a finite cover by affinoid subspaces of X.

Proof. (i) $\implies$ (ii) $\implies$ (iii) is obvious. To prove (iii) we note that $\bigcup_{i\in\mathcal{I}} \bigcup_{j,j'\in\mathcal{J}(i)} V_j \times_{U_i} V_{j'}$ is a covering of $X \times_Y X$ by affinoid subspaces and the restriction of Δ_f is $V_j \times_X V_{j'} \to V_j \times_{U_i} V_{j'}$, which is quasi-compact, since affinoids are quasi-separated. Hence, we may conclude by Lemma 3.1.32(iii). □

Lemma 3.1.34. *Let f and g be composable morphisms of* dRig*. If fg is quasi-separated then so is g.*

Proof. Let us write $g : Z \to Y$ and $f : Y \to X$. Choose a covering $\mathcal{W}$ of X by affinoid subspaces and a covering $\mathcal{U}$ of Y by affinoid subspaces which refines the pullback of $\mathcal{W}$ along f. For each $U \in \mathcal{U}$ let V, V' be affinoid subspaces of Z mapping into U, and say that U maps into the affinoid subspace $W \in \mathcal{W}$. By quasi-separatedness of fg, the morphism $V \times_Z V' \to V \times_X V' = V \times_W V'$ is quasi-compact, and the latter is affinoid. Therefore $V \times_Z V'$ is a finite union of affinoid subspaces of Z. Since such V, V' cover Z, we may appeal to Lemma 3.1.33(iii) to conclude that g is quasi-separated. □

Corollary 3.1.35. *Let f, g be composable morphisms in* dRig*. If f and fg are both qcqs then so is g.*

The following facts were used in §2.3.2, but for the purposes of this section it will be helpful to be more verbose. Recall that we have a diagram of ∞-categories

$$\mathsf{dRig} \hookrightarrow \mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd}) \underset{L}{\overset{\hookrightarrow}{\leftarrow}} \mathrm{Psh}(\mathsf{dAfnd}) \overset{j}{\hookleftarrow} \mathsf{dAfnd} \tag{3.49}$$

where L stands for sheafification [Lur09b] and j is the ∞-categorical Yoneda embedding. In particular for any ∞-category $\mathscr{D}$ admitting small limits we obtain functors

$$\mathrm{Shv}(\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd}), \mathscr{D}) \hookrightarrow \mathrm{Fun}_0(\mathrm{Psh}(\mathsf{dAfnd})^{\mathsf{op}}, \mathscr{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathsf{dAfnd}^{\mathsf{op}}, \mathscr{D}) \tag{3.50}$$

where Fun_0 denotes the limit-preserving functors; the first morphism is induced by precomposition with L and the second is induced by precomposition with j. Here we have equipped $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})$ with the effective epimorphism topology. The image of this composite is $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd}, \mathscr{D})$, c.f. [Lur18b, Proposition 1.3.1.7].

For our purposes this means the following. In Lemma 3.1.16 we proved that the functor QCoh belongs to $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd}, \mathsf{CAlg}(\mathsf{Pr}^L_{\mathsf{st}}))$. We also proved that the weak topology on dAfnd is subcanonical. Hence, by the above procedure we obtain a functor

$$\mathrm{QCoh} : \mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})^{\mathsf{op}} \to \mathsf{CAlg}(\mathsf{Pr}^L_{\mathsf{st}}) \tag{3.51}$$

which is a sheaf in the effective epimorphism topology and which extends QCoh on dAfnd.

Remark 3.1.36. *(i) Chasing the definitions, and using that the weak topology is subcanonical, we see that for $X \in \mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd}^{\mathsf{op}})$ one has*

$$\mathrm{QCoh}(X) \xrightarrow{\sim} \lim_{Y \in \mathsf{dAfnd}^{\mathsf{op}}_{/X}} \mathrm{QCoh}(Y). \tag{3.52}$$

In particular the functor QCoh on $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd}^{\mathsf{op}})$ is right Kan extended from the full subcategory $\mathsf{dAfnd}^{\mathsf{op}}$.

(ii) We can also see from (3.50) *that the functor* QCoh *of* (3.51) *commutes with all small limits. That is, if $(X_k)_{k \in \mathcal{K}}$ is a diagram in $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})$, then the canonical morphism*

$$\mathrm{QCoh}\left(\operatorname*{colim}_{k \in \mathcal{K}} X_k\right) \to \lim_{k \in \mathcal{K}} \mathrm{QCoh}(X_k). \tag{3.53}$$

is an equivalence.

It is clear from Definition 3.1.20 that sheaves on $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})$ restrict to sheaves on dRig in the strong topology. In particular we obtain a functor

$$\mathrm{QCoh} : \mathsf{dRig}^{\mathsf{op}} \to \mathsf{CAlg}(\mathsf{Pr}^L_{\mathsf{st}}) \tag{3.54}$$

which is a sheaf on dRig in the strong topology and extends QCoh on dAfnd. For $X \in \mathsf{dRig}$ we write $\widehat{\otimes}_X$ for the monoidal structure[10] on $\mathrm{QCoh}(X)$. For each morphism $f : X \to Y$ in dRig we denote the induced pullback functor by

$$f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X). \tag{3.55}$$

By construction this is symmetric monoidal and colimit-preserving. Since the categories are presentable this admits a right adjoint denoted by

$$f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y). \tag{3.56}$$

Since f^* is symmetric-monoidal and left adjoint to f_* there is a canonical morphism

$$f_* \widehat{\otimes}_Y \mathrm{id} \to f_*(\mathrm{id} \widehat{\otimes}_X f^*). \tag{3.57}$$

If we are given a Cartesian square in dRig:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & \ulcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array} \tag{3.58}$$

the fact that pullbacks are compatible with composition and left adjoint to pushforwards implies that there is a Beck-Chevalley transformation

$$g^* f_* \to f'_* g'^{,*}. \tag{3.59}$$

Lemma 3.1.37. *With notations as above, assume that the morphism $f : X \to Y$ is* qcqs, *c.f. Definition 3.1.29. Then:*

(i) (Base change). The morphism (3.59) *is an equivalence, for any $g : Y' \to Y$.*

(ii) (Projection formula). The morphism (3.57) *is an equivalence.*

(iii) The functor f_* commutes with all colimits.

Before proving Lemma 3.1.37, we prove an intermediate Lemma.

Lemma 3.1.38. *Let $f : X \to Y$ be a qcqs morphism in* dRig. *Then f* commutes with restrictions, *that is, if $g : Y' \to Y$ is an analytic subspace of Y, then the Beck-Chevalley morphism* (3.59) *is an equivalence.*

Proof. The proof is exactly the same as [Man22, Lemma 2.4.16]; we reproduce it here for the reader's convenience. We proceed in stages.

Step 1: First assume that X and Y are both derived affinoids. One chooses a cover $\{V'_j \to Y'\}_{j \in \mathscr{J}}$ of Y' by rational subspaces of Y. Let $\mathcal{J}$ be the family of finite nonempty subsets of $\mathscr{J}$ and for each $J \in \mathcal{J}$ set $V'_J := \bigcap_{j \in J} V'_j$ and $U'_J := X' \times_{Y'} V'_J$. Set $g_J : V'_J \to Y$

[10] We have chosen to suppress the fact that this is given by the derived tensor product for X affinoid.

and $g'_J : U'_J \to X$ to be the restrictions and let $f'_J : U'_J \to V'_J$ be the base-change of f'. Each U'_J and V'_J is affinoid, and hence by descent and Lemma 3.1.14 one has

$$g^* f_* \simeq \lim_{J \in \mathcal{J}} g^*_J f_* \simeq \lim_{J \in \mathcal{J}} f'_{J,*} g'^{,*}_J \simeq f'_* g'^{,*}, \tag{3.60}$$

proving the Lemma in this case.

Step 2: Now assume that Y is affinoid and X is an analytic subspace of an affinoid space Z. Let $\{U_i \to X\}_{i=1}^n$ be a covering by rational subspaces of Z and let $\mathcal{I}$ be the family of all finite nonempty subsets $I \subseteq \{1, \ldots, n\}$. For $I \in \mathcal{I}$ we set $U_I := \bigcap_{i \in I} U_i$. According to Remark 3.1.18(i) then $f_* \xrightarrow{\sim} \lim_{I \in \mathcal{I}} f_{I,*}$ where $f_I : U_I \to X$ is the restriction. Using that g^* commutes with finite limits, since we are working with stable ∞-categories, one deduces base-change in this case from Step 1.

Step 3: Now assume that Y is affinoid and X is arbitrary. Choose a finite covering $\{U_i \to X\}_{i=1}^n$ by affinoid subspaces, let $\mathcal{I}$ be as before and for each $I \in \mathcal{I}$ set $U_I := \bigcap_{i \in I} U_i$. This is an analytic subspace of an affinoid space and we again have $f_* \xrightarrow{\sim} \lim_{I \in \mathcal{I}} f_{I,*}$. Again using that g^* commutes with finite limits, one deduces base-change in this case from Step 2.

Step 4: Now assume that Y is an analytic subspace of an affinoid space Z and X is arbitrary. One chooses a cover $\{V_j \to Y\}_{j \in \mathscr{J}}$ of Y by rational subspaces of Z. Let $\mathcal{J}$ be the family of finite nonempty subsets of $\mathscr{J}$ and for each $J \in \mathcal{J}$ set $V_J := \bigcap_{j \in J} V_j$ and $U_J := X \times_Y V_J$ and let $f_J : U_J \to V_J$ be the base-change of f. Each V_J is affinoid and hence by Step 3 each f_J commutes with restrictions. Therefore for each Cartesian section $(M_J)_{J \in \mathcal{J}} \in \lim_{J \in \mathcal{J}} \mathrm{QCoh}(U_J) \simeq \mathrm{QCoh}(X)$ one obtains a Cartesian section $(f_{J,*} M_J)_{J \in \mathcal{J}} \in \lim_{J \in \mathcal{J}} \mathrm{QCoh}(V_J) \simeq \mathrm{QCoh}(Y)$. The functor obtained in this way agrees with f_*. In particular, since the covering of Y was arbitrary this implies (by definition of a Cartesian section), that f_* commutes with restrictions.

Step 5: For arbitrary X, Y one can take a cover $\{V_j \to Y\}_{j \in \mathcal{J}}$ of Y by affinoid subspaces. Let $\mathcal{J}$ be the family of finite subsets of $\mathscr{J}$ and for each $J \in \mathcal{J}$ set $V_J := \bigcap_{j \in J} V_j$ and $U_J := X \times_Y V_J$ and let $f_J : U_J \to V_J$ be the base-change of f. Each V_J is an analytic subspace of an affinoid space and hence by Step 4 each f_J commutes with restrictions. Hence by the same reasoning as in Step 4, f_* commutes with restrictions. ☐

Proof of Lemma 3.1.37. (i): This is proved in exactly the same way as [Man22, Proposition 2.4.21]. We reproduce the proof here for the reader's convenience. We proceed in steps.

Step 1: Take affinoid coverings $\mathcal{U}$ of Y and $\mathcal{U}'$ of Y' such that $\mathcal{U}'$ refines the pullback of $\mathcal{U}$ along g. For each $U' \in \mathcal{U}$ let $t_{U'} : U' \to Y'$ be the inclusion. By descent, the collection of pullback functors $\{t^*_{U'} : U' \in \mathcal{U}'\}$ is conservative. Hence it suffices to check (3.59) after applying each $t^*_{U'}$. By the commutation with restrictions proven in Lemma 3.1.38, this reduces the proof of the Lemma to the case when Y and Y' are both affinoid. In the remainder of the proof we will make this assumption. In particular this implies that X and X' are both qcqs.

Step 2: Suppose that X is an analytic subspace of an affinoid space Z. Choose a finite covering $\{U_i \to X\}_{i=1}^n$ of X by rational subspaces of Z. Let $\mathcal{I}$ be the family of finite nonempty subsets of $\{1, \ldots, n\}$. For each $I \in \mathcal{I}$ set $U_I := \bigcap_{j \in I} U_j$ and $U'_I := U_I \times_Y Y'$. By descent one has $f_* \xrightarrow{\sim} \lim_{I \in \mathcal{I}} f_{I,*}$ and $f'_* \xrightarrow{\sim} \lim_{I \in \mathcal{I}} f'_{I,*}$. Each U_I and U'_I is affinoid. Hence, using that g^* commutes with finite limits, and Lemma 3.1.14, one has

$$g^* f_* \simeq \lim_{I \in \mathcal{I}} g^* f_{I,*} \simeq \lim_{I \in \mathcal{I}} f'_{I,*} g'^{,*} \simeq f'_* g'^{,*}, \tag{3.61}$$

proving the Lemma in this case.

Step 3: Suppose now that X is arbitrary. Choose a finite covering $\{U_i \to X\}_{i=1}^n$ of X by affinoid subspaces. Let $\mathcal{I}$ be the family of finite nonempty subsets of $\{1, \ldots, n\}$. For each $I \in \mathcal{I}$ set $U_I := \bigcap_{j \in I} U_j$ and $U'_I := U_I \times_Y Y'$. By descent one has $f_* \xrightarrow{\sim} \lim_{I \in \mathcal{I}} f_{I,*}$ and $f'_* \xrightarrow{\sim} \lim_{I \in \mathcal{I}} f'_{I,*}$. Each U_I and U'_I is an analytic subspace of an affinoid space. Hence, using that g^* commutes with finite limits, and Step 2, one has

$$g^* f_* \simeq \lim_{I \in \mathcal{I}} g^* f_{I,*} \simeq \lim_{I \in \mathcal{I}} f'_{I,*} g'^{,*} \simeq f'_* g'^{,*}, \tag{3.62}$$

proving the Lemma in this case.

(ii): Take affinoid coverings $\mathcal{V}$ of Y and $\mathcal{U}$ of X such that $\mathcal{U}$ refines the pullback of $\mathcal{V}$ along f. For each $U' \in \mathcal{U}$ let $t_U : U \to X$ be the inclusion. By descent the collection of pullback functors $\{t_U^* : U \in \mathcal{U}\}$ is conservative. Hence it suffices to check that (3.57) is an equivalence after applying each t_U^*. By the commutation with restrictions proven in Lemma 3.1.38, and using that pullback functors are symmetric-monoidal, this then reduces the Lemma to the case when X and Y are both affinoids, which is Lemma 3.1.14.

(iii): Let $(M_k)_{k \in \mathcal{K}}$ be a diagram in $\mathrm{QCoh}(X)$. We need to show that the canonical morphism

$$\operatorname*{colim}_{k \in \mathcal{K}} f_* M_k \to f_* \operatorname*{colim}_{k \in \mathcal{K}} M_k, \tag{3.63}$$

is an equivalence. Using the same notations as in the proof of (ii), it suffices to check that (3.63) is an equivalence after applying each t_U^*. By the commutation with restrictions proven in Lemma 3.1.38, and using that each t_U^* is colimit-preserving (it is a left adjoint), one reduces to the case when X and Y are both affinoids, which follows from Lemma 3.1.14. ☐

Corollary 3.1.39. *The functor* QCoh *extends to a six-functor formalism*

$$\mathrm{QCoh} : \mathrm{Corr}(\mathsf{dRig}, \mathrm{qcqs}) \to \mathsf{Pr}_{\mathsf{st}}^{L,\otimes}. \tag{3.64}$$

In this six-functor formalism every qcqs morphism f satisfies $f_! = f_*$.

Proof. This is immediate from Lemma 3.1.37 and [Man22, Proposition A.5.10]. To be clear, in the language of *loc. cit.* we take the *suitable decomposition* to be $(I, P) = (\mathrm{equivalences}, \mathrm{qcqs})$. ☐

3.1.5 Six-functor formalism in derived rigid geometry

We will apply the formalism of §2.3.2 in the following set-up (with notations as in that section):

- ⋆ We take $\mathscr{V} := D(\mathsf{CBorn}_K)$, so that $\mathscr{E} := \mathsf{CAlg}(D(\mathsf{CBorn}_K))^{\mathsf{op}}$ and we consider $\mathsf{dAfnd} \subseteq \mathscr{E}$. We take $\tau :=$ weak to be the weak topology on dAfnd.

By Lemma 3.1.2, the assumptions of §2.3.2 are satisfied; we obtain a six-functor formalism

$$\mathrm{QCoh} : \mathrm{Corr}(\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd}), E)^{\otimes} \to \mathsf{Pr}_{\mathsf{st}}^{L,\otimes}, \tag{3.65}$$

with the following properties:

- ⋆ The class $E \supseteq \mathrm{rep}$ is stable under disjoint unions, $*$-local on the target, !-local on the source, is tame, and satisfies $E \subseteq \delta E$.

⋆ Every morphism $f \in \text{rep}$ satisfies $f_! \simeq f_*$ (Corollary 2.3.16).

The purpose of this section is to prove the following Theorem:

Theorem 3.1.40. *In the six-functor formalism* (3.65)*, every qcqs morphism $f : X \to Y$ of derived rigid varieties belongs to the class E. The restriction of this six-functor formalism along* $\mathrm{Corr}(\mathsf{dRig}, \mathrm{qcqs}) \to \mathrm{Corr}(\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd}), E)$ *is equivalent to the six-functor formalism constructed in Corollary 3.1.39. In particular, for every qcqs morphism $f : X \to Y$ between objects of* dRig *there is a canonical equivalence $f_! \simeq f_*$.*

The proof is postponed to the end of this subsection. Essentially, we want to prove that the six-functor formalism QCoh extends uniquely from $(\mathsf{dAfnd}, \mathrm{all})$ to $(\mathsf{dRig}, \mathrm{qcqs})$.

Lemma 3.1.41. *Let $X \in \mathsf{dRig}$, and let $\{U_i \to X\}_{i=1}^n$ be a finite covering by analytic subspaces. Let $\mathcal{I}$ be the family of finite nonempty subsets of $\{1, \dots, n\}$ and for each $I \in \mathcal{I}$ set $U_I := \bigcap_{i \in I} U_i$. Assume that for each $I \in \mathcal{I}$, the morphism $U_I \to X$ is quasi-compact*[11]*. Then the canonical morphism*

$$\mathrm{QCoh}^!(X) \to \lim_{I \in \mathcal{I}} \mathrm{QCoh}^!(U_I) \tag{3.66}$$

is an equivalence. Here we are using the six-functor formalism from Corollary 3.1.39.

Proof. Let $t_I : U_I \to X$ be the inclusions. Since QCoh^* is a sheaf, we know that there is an equivalence of categories

$$\mathrm{QCoh}^*(X) \to \lim_{I \in \mathcal{I}} \mathrm{QCoh}^*(U_I) \tag{3.67}$$

where the functor from left to right sends $M \to (t_I^* M)_I$ and the functor from right to left sends $(M_I)_I \to \lim_I t_{I,*} M_I$. In particular the counit $\mathrm{id}_X \to \lim_I t_{I,*} t_I^*$ is an equivalence. In order to prove the Lemma we need to show that $(t_I^!)_I$ induces an equivalence

$$\mathrm{QCoh}^!(X) \to \lim_{I \in \mathcal{I}} \mathrm{QCoh}^!(U_I). \tag{3.68}$$

There is a natural adjunction in which this functor is right adjoint to $\mathrm{colim}_I\, t_{I,!}$. We note that all the t_I are qcqs and therefore $t_{I,!} = t_{I,*}$.

Firstly, we will check that the counit is an equivalence. By the previous, we know that $1_X \xrightarrow{\sim} \lim_I t_{I,*} 1_{U_I}$. Let $M \in \mathrm{QCoh}(X)$. We have equivalences

$$\begin{aligned} M &\simeq \underline{\mathrm{Hom}}_X(1_X, M) \\ &\simeq \mathrm{colim}_I \underline{\mathrm{Hom}}_X(t_{I,*} 1_{U_I}, M) \\ &\simeq \mathrm{colim}_I t_{I,*} \underline{\mathrm{Hom}}_{U_I}(1_{U_I}, t_I^! M) \\ &\simeq \mathrm{colim}_I t_{I,*} t_I^! M. \end{aligned} \tag{3.69}$$

Here, in the second line we used the property of stable ∞-categories, and in the third line we used the identity which is adjoint to the projection formula, c.f. Remark 2.3.4(ii). Therefore the counit is an equivalence. To check that the unit is an equivalence we need to show that, given $(M_I)_I$, for each $J \in \mathcal{I}$ the natural morphism $M_J \to t_J^! \,\mathrm{colim}_I\, t_{I,*} M_I$ is an equivalence. Since the relevant ∞-categories are stable, we can exchange $t_I^!$ with this finite colimit, and then use base-change, whence the covering becomes split and the morphism is obviously an equivalence. □

[11] It is automatically quasi-separated, being a monomorphism.

Corollary 3.1.42. *With notations as in Lemma 3.1.41. Set* $Y := \coprod_{i=1}^n U_i$*. Then, the canonical morphism*

$$\mathrm{QCoh}^!(X) \to \lim_{[m]\in\Delta} \mathrm{QCoh}^!(Y^{m+1/X}) \tag{3.70}$$

is an equivalence. In particular $t := \coprod_{i=1}^n t_i : Y \to X$ *is of universal* $!$*-descent.*

Proof. This is quite similar to [Man22, Proposition 2.6.3] which is a consequence of the Lurie-Beck-Chevalley condition [Lur17, Corollary 4.7.5.3].

We first check condition (1) in [Lur17, Corollary 4.7.5.3]. Let $(M_m)_{[m]\in\Delta_+^{\mathrm{op}}}$ be a $t^!$-split simplicial object in $\mathrm{QCoh}(X)$. In particular, it is $t_* t^!$-split and in fact $t_{I,*}t_I^!$-split for every $I \in \mathcal{I}$. This implies that $(\mathrm{sk}_k\, t_{I,*}t_I^! M_\bullet)_{k\geqslant 0}$ is a constant Ind-object. By dual arguments to [Mat16, Proposition 3.10], it then follows that $(\mathrm{sk}_k\, \mathrm{colim}_I\, t_{I,*}t_I^! M_\bullet)_{k\geqslant 0}$ is a constant Ind-object. By Lemma 3.1.41 above this is equivalent to the Ind-object $(\mathrm{sk}_k\, M_\bullet)_{k\geqslant 0}$. In this case it is obvious that $t^!$ commutes with the geometric realization of $M_\bullet$, since the relevant ∞-categories are stable.

Now condition (2) in [Lur17, Corollary 4.7.5.3] follows from base-change, and conservativity of $t^!$ follows from Lemma 3.1.41. Therefore, the result of *loc. cit.* gives the desired equivalence. □

Definition 3.1.43. *A morphism* $f : X \to Y$ *in* $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})$ *is called* semi-separated *if the diagonal* $\Delta_f : X \to X \times_Y X$ *belongs to the class* rep *of representable morphisms. An object* $X \in \mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})$ *is called* semi-separated *if the morphism* $X \to *$ *is semi-separated.*

Proposition 3.1.44. *The six-functor formalism* QCoh *extends uniquely from* $(\mathsf{dAfnd}, \mathrm{all})$ *to* $(\mathsf{dRig}, \mathrm{qcqs})$*.*

Proof. By [Man22, Proposition A.5.16] the six-functor formalism QCoh extends uniquely from $(\mathsf{dAfnd}, \mathrm{all})$ to $(\mathsf{qcssdRig}, \mathrm{rep})$, where $\mathsf{qcssdRig}$ denotes the category of quasi-compact and *semi-separated* derived rigid spaces, i.e., those having representable diagonal. Now Corollary 3.1.42 together with [Man22, Proposition A.5.14] implies that QCoh further extends uniquely to a six-functor formalism on $(\mathsf{qcssdRig}, \mathrm{all})$. To be clear, for the application of [Man22, Proposition A.5.14] here, one takes the class S of *special covers* (in the language of *loc. cit.*) to be the class of finite coverings by affinoids.

Now [Man22, Proposition A.5.16] again implies that the six-functor formalism QCoh extends uniquely from $(\mathsf{qcssdRig}, \mathrm{all})$ to $(\mathsf{qcqsdRig}, F)$ where F is the collection of edges which are representable in $\mathsf{qcssdRig}$. Now Corollary 3.1.42 again with [Man22, Proposition A.5.14] implies that QCoh further extends uniquely to a six-functor formalism on $(\mathsf{qcqsdRig}, \mathrm{all})$ (again, for the application of [Man22, Proposition A.5.14], one takes the class S of *special covers* to be the class of finite coverings by affinoids).

Finally, an application of [Man22, Proposition A.5.16] implies that QCoh extends uniquely from $(\mathsf{qcqsdRig}, \mathrm{all})$ to $(\mathsf{dRig}, \mathrm{qcqs})$. □

Proposition 3.1.45. *In the six-functor formalism* (3.51)*, the class* E *contains all qcqs morphisms between derived rigid spaces.*

Proof. Since the class E is $*$*-local on the target* one reduces immediately to the case when $f : X \to Y$ is a qcqs morphism with Y affinoid.

We claim that if $g : Z \to W$ is a quasi-compact and *semi-separated* morphism of derived rigid varieties with affinoid target, there is a canonical equivalence $g_! \simeq g_*$ in this six-functor formalism. Indeed, because $\Delta_g \in \mathrm{rep}$, one has $\Delta_{g,!} \simeq \Delta_{g,*}$ and from this one obtains a canonical morphism $g_! \to g_*$, which is the adjoint to the composite

$$g^* g_! \simeq \pi_{1,!}\pi_2^* \to \pi_{1,!}\Delta_{g,*}\Delta_g^*\pi_2^* \simeq \pi_{1,!}\Delta_{g,!}\Delta_g^*\pi_2^* \simeq \mathrm{id}; \tag{3.71}$$

here $\pi_1, \pi_2 : Z \times_W Z \to Z$ are the projections, and we used base-change. Now this morphism $g_! \to g_*$ is an equivalence: by $*$-descent on the source, this follows if g_* satisfies base-change, but we have already proved this in Lemma 3.1.37 above.

Now take a finite cover $\{U_i \to X\}_i$ of X by affinoids; then all the $t_I : U_I \to X$ are quasi-compact[12], hence by the above they satisfy $t_{I,!} \simeq t_{I,*}$, hence same argument as in Lemma 3.1.41 and Corollary 3.1.42 above shows that $\{U_i \to X\}_i$ is of universal !-descent. Since the class E is *!-local on the source*, we conclude. ☐

Proof of Theorem 3.1.40. By Proposition 3.1.45 above, we may consider the restriction from $(\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd}), E)$ to $(\mathsf{dRig}, \mathrm{qcqs})$. Now the Theorem follows immediately from the unicity proved in Proposition 3.1.44 together with Corollary 3.1.39. ☐

The results of this subsection, together with §2.3.3, can be used to obtain a theory of Fourier–Mukai transforms in derived rigid geometry.

Proposition 3.1.46. *Let $f : Z \to Y$ be a quasi-compact and semi-separated morphism of derived rigid spaces. Then f is transformable in the sense of Definition 2.3.21.*

Proof. One immediately reduces to the case when $Y = \mathrm{dSp}(B)$ is a derived affinoid and hence Z is quasi-compact and semi-separated. Let $\{U_i\}_{i=1}^n$ be a finite cover of Z by affinoid subsheaves. Then $\coprod_{i=1}^n U_i$ is affinoid. In Corollary 3.1.42 above, it was proven that $\coprod_{i=1}^n U_i \to Z$ is of universal !-descent. Therefore f is transformable. ☐

Corollary 3.1.47. *Let Y be a semi-separated derived rigid space, let $f : Z \to Y$ be a quasi-compact and semi-separated morphism of derived rigid spaces, and let $X \to Y$ be an arbitrary morphism in $\mathrm{Shv}_{\mathrm{weak}}(\mathsf{dAfnd})$. Then the Fourier-Mukai transform gives an equivalence of ∞-categories*

$$FM : \mathrm{QCoh}(Z \times_Y X) \xrightarrow{\sim} \mathrm{Fun}^L_{\mathrm{QCoh}(Y)}(\mathrm{QCoh}(Z), \mathrm{QCoh}(X)), \tag{3.72}$$

and an equivalence of monoidal ∞-categories

$$FM : \mathrm{QCoh}(Z \times_Y Z) \xrightarrow{\sim} \mathrm{Fun}^L_{\mathrm{QCoh}(Y)}(\mathrm{QCoh}(Z), \mathrm{QCoh}(Z)). \tag{3.73}$$

Proof. Given Proposition 3.1.46, this follows from Theorem 2.3.22. ☐

3.1.6 Infinite covers of universal !-descent

In the remainder of this section, we introduce the following notation. Let $X \in \mathsf{dRig}$ and let $S \subseteq |X|$ be a closed subset of the underlying topological space of X. Set $V := |X| \setminus S \subseteq |X|$, which we may identify with an analytic subspace $j : V \hookrightarrow X$. We define full subcategories

$$\Gamma_S \,\mathrm{QCoh}(X) \subseteq \mathrm{QCoh}(X) \quad \text{and} \quad \mathrm{L}_S \,\mathrm{QCoh}(X) \subseteq \mathrm{QCoh}(X), \tag{3.74}$$

[12] And semi-separated because they are monomorphisms.

as the full subcategories spanned by objects M such that $j^*M \simeq 0$, and $j^!M \simeq 0$, respectively. If we assume that j is quasi-compact, so that $j_! = j_*$, then base-change implies that the inclusions of these full subcategories admit adjoints given by

$$\Gamma_S := \mathrm{Fib}(\mathrm{id} \to j_* j^*) \quad \text{and} \quad \mathrm{L}_S := \mathrm{Cofib}(j_! j^! \to \mathrm{id}). \tag{3.75}$$

To be clear, Γ_S is right adjoint to the inclusion $\Gamma_S \,\mathrm{QCoh}(X) \subseteq \mathrm{QCoh}(X)$ and L_S is left adjoint to the inclusion $\mathrm{L}_S \,\mathrm{QCoh}(X) \subseteq \mathrm{QCoh}(X)$. We note that Γ_S is colimit-preserving and L_S is limit-preserving.

Lemma 3.1.48. *Let $X \in \mathsf{dRig}$. Suppose that we are given sequences $V_1 \supseteq V_2 \supseteq V_3 \supseteq \dots$ and $U_1 \subseteq U_2 \subseteq U_3 \subseteq \dots$ of analytic subspaces of X such that:*

- ⋆ *For every $n \geqslant 1$ the morphisms $V_n \to X$ and $U_n \to X$ are quasi-compact and one has $U_n \cup V_n = X$ and $U_n \cap V_{n+1} = \emptyset$.*

Let us write $S_n := |X| \setminus V_n$ and $\Gamma_n \,\mathrm{QCoh}(X) := \Gamma_{S_n} \,\mathrm{QCoh}(X)$. Then, there is an equivalence of Pro-systems

$$\text{“}\lim_n\text{”}\, \mathrm{QCoh}^!(U_n) \simeq \text{“}\lim_n\text{”}\,\Gamma_n \,\mathrm{QCoh}(X), \tag{3.76}$$

which is compatible with the maps from $\mathrm{QCoh}(X)$.

Proof. Let us write $j_n : V_n \to X$ and $t_n : U_n \to X$ for the inclusions. Let us also abbreviate $\Gamma_n := \Gamma_{S_n}$ and write incl_n for the inclusion of the full subcategory $\Gamma_n \,\mathrm{QCoh}(X) \subseteq \mathrm{QCoh}(X)$. Consider the diagram

$$\cdots \leftarrow \Gamma_n \,\mathrm{QCoh}(X) \xleftarrow{\Gamma_n t_{n,*}} \mathrm{QCoh}(U_n) \xleftarrow{t_n^! \,\mathrm{incl}_{n+1}} \Gamma_{n+1} \,\mathrm{QCoh}(X) \xleftarrow{\Gamma_{n+1} t_{n+1,*}} \mathrm{QCoh}(U_{n+1}) \leftarrow \dots \tag{3.77}$$

We need to show that there are natural equivalences

$$t_n^! \,\mathrm{incl}_{n+1} \Gamma_{n+1} t_{n+1,*} \simeq t_{n,n+1}^! \tag{3.78}$$

and

$$\Gamma_n t_{n,*} t_n^! \,\mathrm{incl}_{n+1} \simeq \Gamma_{n,n+1}. \tag{3.79}$$

In order to do this, we first observe that the left adjoint to $t_n^! \,\mathrm{incl}_{n+1}$ is given by $t_{n,*}$ since this already factors through $\Gamma_{n+1} \,\mathrm{QCoh}(X)$: by base-change one has $j_{n+1}^* t_{n,*} \simeq 0$ as $U_n \cap V_{n+1} = \emptyset$. Therefore, by passing to adjoints, it is enough to show that

$$t_{n+1}^* \,\mathrm{incl}_{n+1} \, t_{n,*} \simeq t_{n,n+1,*} \tag{3.80}$$

and

$$t_{n,*} t_n^* \,\mathrm{incl}_n \simeq \mathrm{incl}_{n,n+1}\,. \tag{3.81}$$

The equivalence (3.80) is a consequence of base-change. The equivalence (3.81) is a consequence of descent applied to the covering $X = V_n \cup U_n$. □

Lemma 3.1.49. *Let $X \in \mathsf{dRig}$. Suppose that we are given sequences $V_1 \supseteq V_2 \supseteq V_3 \supseteq \dots$ and $U_1 \subseteq U_2 \subseteq U_3 \subseteq \dots$ of analytic subspaces of X such that:*

- ⋆ *For every $n \geqslant 1$ the morphisms $V_n \to X$ and $U_n \to X$ and $U_n \cap V_n \to X$ are quasi-compact, and one has $U_n \cup V_n = X$ and $U_n \cap V_{n+1} = \emptyset$.*

Let us write $S_n := |X| \setminus V_n$ *and* $\mathrm{L}_n \operatorname{QCoh}(X) := \mathrm{L}_{S_n} \operatorname{QCoh}(X)$*. Then, there is an equivalence of Pro-systems*

$$\text{“}\lim_n\text{”} \operatorname{QCoh}^*(U_n) \simeq \text{“}\lim_n\text{”} \mathrm{L}_n \operatorname{QCoh}(X), \tag{3.82}$$

which is compatible with the maps from $\operatorname{QCoh}(X)$*.*

Proof. Let us write $j_n : V_n \to X$ and $t_n : U_n \to X$ for the inclusions. Let us also abbreviate $\mathrm{L}_n := \mathrm{L}_{S_n}$ and write incl'_n for the inclusion of the full subcategory $\mathrm{L}_n \operatorname{QCoh}(X) \subseteq \operatorname{QCoh}(X)$. Consider the diagram

$$\cdots \leftarrow \mathrm{L}_n \operatorname{QCoh}(X) \xleftarrow{\mathrm{L}_n\, t_{n,*}} \operatorname{QCoh}(U_n) \xleftarrow{t_n^* \,\mathrm{incl}'_{n+1}} \mathrm{L}_{n+1} \operatorname{QCoh}(X) \xleftarrow{\mathrm{L}_{n+1}\, t_{n+1,*}} \operatorname{QCoh}(U_{n+1}) \leftarrow \dots \tag{3.83}$$

We need to show that there are natural equivalences

$$t_n^* \,\mathrm{incl}'_{n+1} \mathrm{L}_{n+1}\, t_{n+1,*} \simeq t^*_{n,n+1} \tag{3.84}$$

and

$$\mathrm{L}_n\, t_{n,*} t_n^* \,\mathrm{incl}'_{n+1} \simeq \mathrm{L}_{n,n+1}\,. \tag{3.85}$$

In order to do this, we first observe that the right adjoint to $t_n^* \,\mathrm{incl}'_{n+1}$ is given by $t_{n,*}$, since this already factors through $\mathrm{L}_{n+1} \operatorname{QCoh}(X)$: base-change one has $j^!_{n+1} t_{n,*} \simeq 0$ as $U_n \cap V_{n+1} = \emptyset$. Therefore, by passing to adjoints, it is enough to show that

$$t^!_{n+1} \,\mathrm{incl}'_{n+1}\, t_{n,*} \simeq t_{n,n+1,*} \tag{3.86}$$

and

$$t_{n,*} t_n^! \,\mathrm{incl}'_n \simeq \mathrm{incl}'_{n,n+1}\,. \tag{3.87}$$

The equivalence (3.86) is an consequence of base-change. The equivalence (3.87) is a consequence of Lemma 3.1.41 applied to the covering $X = V_n \cup U_n$. ☐

Proposition 3.1.50. *Let* $X \in \mathsf{dRig}$*. Suppose that we are given sequences* $V_1 \supseteq V_2 \supseteq V_3 \supseteq \dots$ *and* $U_1 \subseteq U_2 \subseteq U_3 \subseteq \dots$ *of analytic subspaces of* X *such that,*

- ⋆ *For every* $n \geqslant 1$ *the morphisms* $V_n \to X$ *and* $U_n \to X$ *are quasi-compact and one has* $U_n \cup V_n = X$ *and* $U_n \cap V_{n+1} = \emptyset$*.*
- ⋆ *One has* $\bigcup_{n \geqslant 1} U_n = X$*.*

Then, the natural morphism $\operatorname{QCoh}^!(X) \to \lim_n \operatorname{QCoh}^!(U_n)$ *is an equivalence.*

Proof. We use notations as in Lemma 3.1.48. We claim that the canonical morphism

$$\operatorname{QCoh}(X) \to \lim_n \Gamma_n \operatorname{QCoh}(X) \tag{3.88}$$

sending $M \mapsto (\Gamma_n M)_n$, is an equivalence. This functor is right adjoint to the functor sending $(M_n)_n \mapsto \operatorname{colim}_n M_n$. Let $M \in \operatorname{QCoh}(X)$. We first check that the counit morphism

$$\operatorname*{colim}_n \mathrm{incl}_n \Gamma_n M \to M \tag{3.89}$$

is an equivalence. By descent, it suffices to check this after applying t_m^* for each $m \geqslant 0$. By using that t_m^* commutes with colimits, and that $U_m \cap V_{m+1} = \emptyset$, together with base-change, we see that (3.89) is indeed an equivalence after applying t_m^*. Now we check the unit morphism. We need to show that for each Cartesian section $(M_n)_n$ belonging to the right side of (3.88), and each $m \geqslant 0$, the morphism

$$M_m \to \Gamma_m\big(\operatorname*{colim}_n \operatorname{incl}_n M_n\big), \tag{3.90}$$

is an equivalence. This follows from the fact that Γ_m commutes with colimits. Therefore (3.88) is an equivalence, and we can appeal to Lemma 3.1.48 to deduce the statement of the Proposition. ☐

Corollary 3.1.51. *With notations as in Proposition 3.1.50. Set $Y := \coprod_{n \geqslant 1} U_n$. Then, the canonical morphism*

$$\mathrm{QCoh}^!(X) \to \lim_{[m]\in\Delta} \mathrm{QCoh}^!(Y^{m+1/X}) \tag{3.91}$$

is an equivalence. In particular $Y \to X$ is of universal !-descent.

Proof. By using that coproducts in dRig are universal, and that $\mathrm{QCoh}^!$ commutes with products, this follows from Proposition 3.1.50 and the Bousfield-Kan formula for ∞-categories [MG19]. ☐

Example 3.1.52. *Let $\varrho > 1, \varrho \in |K^\times|$ be an element of the valuation group of K, and let $X := \mathbf{A}^1_K := \operatorname{colim}_n \mathbf{D}^1_K(\varrho^n)$ be the rigid-analytic affine line, regarded as an object of dRig. Here $\mathbf{D}_K(\varrho^n) := \mathrm{dSp}(K\langle \varrho^{-n} T\rangle)$ is the rigid-analytic disk of radius ϱ^n. With notations as in Proposition 3.1.50, one can take $U_n := \mathbf{D}^1_K(\varrho^{2n})$ and $V_n := \mathbf{A}_{[\varrho^{2n-1},\infty)}$, and the hypotheses of the Proposition are satisfied, where the latter is the rigid-analytic annulus of radius $\geqslant \varrho^{2n-1}$.*

3.1.7 Local cohomology

In this section, we will develop in different generality an idea that was used in §3.1.6. A classical theory of local cohomology for rigid- and complex-analytic varieties was developed by Kisin in [Kis99a].

Let $X \in \mathsf{dRig}$ and let $S \subseteq |X|$ be a closed subset of the underlying topological space of X. Then $U := |X| \setminus S \subseteq |X|$ may be identified with an analytic subspace $j : U \hookrightarrow X$[13]. We define

$$\Gamma_S \, \mathrm{QCoh}(X) \subseteq \mathrm{QCoh}(X) \tag{3.92}$$

as the full subcategory of $\mathrm{QCoh}(X)$ spanned by objects M such that $j^*M \simeq 0$. Since j^* commutes with colimits, it follows that the inclusion incl_S of this full subcategory commutes with colimits, and therefore[14] admits a right adjoint:

$$\mathrm{incl}_S : \Gamma_S \, \mathrm{QCoh}(X) \leftrightarrows \mathrm{QCoh}(X) : \Gamma_S, \tag{3.93}$$

which we have denoted by Γ_S. Now suppose in addition that j belongs to the class E of §3.1.5 so that it is !-able. There is a canonical morphism $j^! \to j^*$, defined as the composite

$$j^! \to j^! j_* j^* \simeq j^*, \tag{3.94}$$

[13]Indeed, from now on, we may deliberately confuse open subsets $U \subseteq |X|$ with analytic subspaces $U \hookrightarrow X$.

[14]Since the categories are presentable.

here the first morphism comes from the unit of $j^* \dashv j_*$ and the second is by base-change. *In this section we will be interested in the situation when the canonical morphism $j^! \to j^*$ is an equivalence.*[15] Under this assumption, $\Gamma_S \operatorname{QCoh}(X)$ may equivalently be described as the full subcategory of $\operatorname{QCoh}(X)$ spanned by objects M such that $j^! M \simeq 0$. Since $j^!$ commutes with limits, it follows that the inclusion incl_S also commutes with limits, and therefore admits a left adjoint:

$$\mathrm{L}_S : \Gamma_S \operatorname{QCoh}(X) \leftrightarrows \operatorname{QCoh}(X) : \mathrm{incl}_S, \tag{3.95}$$

which we have denoted[16] by L_S. Various formal properties of these functors are listed in the next Proposition.

Proposition 3.1.53. *Let $X \in \mathsf{dRig}$ and let $S \subseteq |X|$ be a closed subset. Set $U := |X| \setminus S$ and let $j : U \to X$ be the inclusion of the corresponding analytic subspace. Assume that $j \in E$ and that the canonical morphism $j^! \to j^*$ is an equivalence. Then:*

(i) *There is an equivalence $\Gamma_S \simeq \mathrm{Fib}(\mathrm{id} \to j_* j^*)$.*

(ii) *There are equivalences $\mathrm{L}_S \simeq \mathrm{Cofib}(j_! j^! \to \mathrm{id}) \simeq \mathrm{Cofib}(j_! j^! 1_X \to 1_X) \widehat{\otimes}_X \mathrm{id}$, so that the functor L_S is given by tensoring with the idempotent algebra object $\mathrm{Cofib}(j_! j^! 1_X \to 1_X)$. In particular $\Gamma_S \operatorname{QCoh}(X)$ is symmetric-monoidal, with the monoidal structure inherited from $\operatorname{QCoh}(X)$ and tensor-unit given by $\mathrm{Cofib}(j_! j^! 1_X \to 1_X)$.*

(iii) *In the sequence*

$$\Gamma_S(\operatorname{QCoh}(X)) \xrightarrow{\mathrm{incl}_S} \operatorname{QCoh}(X) \xrightarrow{j^*} \operatorname{QCoh}(U) \tag{3.96}$$

the right adjoints[17] *Γ_S and j_* satisfy $\Gamma_S \,\mathrm{incl}_S \simeq \mathrm{id}$ and $j^* j_* \simeq \mathrm{id}$.*

(iv) *In the sequence*

$$\operatorname{QCoh}(U) \xrightarrow{j_!} \operatorname{QCoh}(X) \xrightarrow{\mathrm{L}_S} \Gamma_S \operatorname{QCoh}(X), \tag{3.97}$$

the composite $\mathrm{L}_S\, j_! \simeq 0$, and the right adjoints $j^!$ and incl_S satisfy $j^! j_! \simeq \mathrm{id}$ and $\mathrm{L}_S \,\mathrm{incl}_S \simeq \mathrm{id}$. Moreover, the right adjoints $j^!$ and incl_S commute with colimits, so that (3.97) *is a split-exact sequence in the sense of [Jia23, Appendix B].*

Proof. (i): Let us temporarily denote $F := \mathrm{Fib}(\mathrm{id} \to j_* j^*)$. We will show that

$$F \,\mathrm{incl}_S \simeq \mathrm{id} \quad \text{and} \quad j^* F \simeq 0. \tag{3.98}$$

By using that $j^! \xrightarrow{\sim} j^*$, and base-change, one has

$$j^* F \simeq \mathrm{Fib}(j^* \to j^* j_* j^*) \simeq \mathrm{Fib}(j^* \to j^! j_* j^*) \simeq \mathrm{Fib}(j^* \to j^*) \simeq 0. \tag{3.99}$$

By definition, we have $j^* \,\mathrm{incl}_S \simeq 0$, so $F \,\mathrm{incl}_S \simeq \mathrm{id}$. We can define a counit $\varepsilon : \mathrm{incl}_S F \to \mathrm{id}$ coming from that canonical morphism $\mathrm{Fib}(\mathrm{id} \to j_* j^*) \to \mathrm{id}$ and a unit morphism $\eta : \mathrm{id} \simeq F \,\mathrm{incl}_S$, and one can verify the zig-zag identities using (3.98). Therefore, by the uniqueness of adjoints, we obtain $\Gamma_S \simeq \mathrm{Fib}(\mathrm{id} \to j_* j^*)$.

[15] Note that this is *not* satisfied for the inclusions $j_n : U_n \to X$ appearing in the proof of Proposition 3.1.50.

[16] I chose this notation because L looks like Γ upside down, and also because the functor is a left adjoint.

[17] Note especially here, that we do not assert that the right adjoints j_* and Γ_S commute with colimits.

(ii): The proof that $\mathrm{L}_S \simeq \mathrm{Cofib}(j_! j^! \to \mathrm{id})$ is quite similar to the proof of (i) and so we omit it. For the second part, it suffices to show that there is an natural equivalence

$$j_! j^! 1_X \widehat{\otimes}_X \mathrm{id} \xrightarrow{\sim} j_! j^!. \tag{3.100}$$

Indeed, one has equivalences $j_! j^! 1_X \widehat{\otimes}_X \mathrm{id} \simeq j_!(j^! 1_X \widehat{\otimes}_U j^*) \simeq j_! j^!$, where the first is the projection formula and the second is because $j^! \simeq j^*$ is symmetric-monoidal. Due to the fact that $\mathrm{incl}_S \Gamma_S$ is an idempotent monad we see that $\mathrm{Cofib}(j_! j^! 1_X \to 1_X)$ is an idempotent algebra object and $\mathrm{incl}_S \Gamma_S$ is given by tensoring with this algebra object. It follows formally from this that $\Gamma_S \operatorname{QCoh}(X)$ is symmetric monoidal with monoidal structure inherited from $\operatorname{QCoh}(X)$ and tensor-unit given by $\mathrm{Cofib}(j_! j^! 1_X \to 1_X)$.[18]

(iii): The only thing to check is that $j^* j_* \to \mathrm{id}$ is an equivalence. As noted above this can be deduced from the equivalence $j^! \xrightarrow{\sim} j^*$ and base-change.

(iv): The identity $\mathrm{L}_S j_! \simeq 0$ follows by passing to left adjoints in $j^! \mathrm{incl}_S \simeq 0$. The formula $j^! j_! \simeq \mathrm{id}$ can be deduced from the equivalence $j^! \xrightarrow{\sim} j^*$ and base-change. □

Let $X \in \mathsf{dRig}$. In the remainder of this section, we will produce examples of inclusions $j : U \to X$ of analytic subspaces satisfying $j^! \xrightarrow{\sim} j^*$, and also give some different formulas for the functors Γ_S and L_S.

Proposition 3.1.54. *Let $X \in \mathsf{dRig}$ and let $S \subseteq |X|$ be a closed subset of the underlying topological space. Set $U := |X| \setminus S \subseteq |X|$. Suppose that we are given sequences $V_1 \supseteq V_2 \supseteq \cdots \supseteq S$ and $U_1 \subseteq U_2 \subseteq \cdots \subseteq U$ of open subsets of $|X|$ such that:*

- ⋆ *For every $n \geqslant 1$ the morphisms $V_n \to X$ and $U_n \to X$ are quasi-compact and one has $U_n \cup V_n = X$ and $U_n \cap V_{n+1} = \emptyset$.*
- ⋆ *One has $\bigcup_{n \geqslant 1} U_n = U$.*

Then:

(i) The morphism $j : U \to X$ belongs to the class E, i.e., it is !-able.

(ii) Let $k_n : V_n \to X$ be the inclusions. Then:

(a) There is an equivalence of functors $\mathrm{colim}_n\, k_{n,*} k_n^* \simeq \mathrm{Cofib}(j_! j^! \to \mathrm{id})$.

(b) There is an equivalence of functors $\lim_n k_{n,*} k_n^! \simeq \mathrm{Fib}(\mathrm{id} \to j_* j^*)$.

(iii) The canonical morphism $j^! \to j^*$ is an equivalence.

Before the proof, we give an example.

Example 3.1.55. *Let $X = \mathrm{dSp}(A) \in \mathsf{dRig}$ be an affinoid and let $I \subseteq \pi_0 A$ be an ideal. Choose generators $f_1, \dots, f_k$ for I. By using the homeomorphism $r : |\operatorname{Spa}(\pi_0 A, (\pi_0 A)^\circ)| \xrightarrow{\sim} |X|$ coming from Remark 3.1.28, we may define open subsets U_n and V_n of $|X|$ by:*

$$\begin{aligned} r^{-1} V_n &:= \{x \in |\operatorname{Spa}(\pi_0 A, (\pi_0 A)^\circ)| : |f_i|_x \leqslant p^{-2n} \text{ for all } 1 \leqslant i \leqslant k\}, \\ r^{-1} U_n &:= \{x \in |\operatorname{Spa}(\pi_0 A, (\pi_0 A)^\circ)| : |f_i|_x \geqslant p^{-(2n+1)} \text{ for some } 1 \leqslant i \leqslant k\}. \end{aligned} \tag{3.101}$$

Then:

[18] This can be regarded as a "categorification" of the following. Let R be a commutative ring and let $e \in R$ be an idempotent. Then $I := \ker(R \to R_e)$ is a ring, isomorphic to $(1-e)R$, with multiplication inherited from R and unit $1 - e$.

(i) The sequences $\{V_n\}_{n\geqslant 1}$ *and* $\{U_n\}_{n\geqslant 1}$ *satisfy the hypotheses of Proposition 3.1.54, with*

$$\begin{aligned} r^{-1}S &= \{x \in |\operatorname{Spa}(\pi_0 A, (\pi_0 A)^\circ)| : |f_i|_x = 0 \textit{ for all } 1 \leqslant i \leqslant k\}, \\ r^{-1}U &= \{x \in |\operatorname{Spa}(\pi_0 A, (\pi_0 A)^\circ)| : |f_i|_x > 0 \textit{ for some } 1 \leqslant i \leqslant k\}. \end{aligned} \tag{3.102}$$

(ii) The sequence $\{V_n\}_{n\geqslant 1}$ *is a cofinal system of open neighbourhoods of* S *in* $|X|$*. Indeed, let* $S \subseteq V' \subseteq |X|$ *be another open subset. Then* $\{V'\} \cup \{U_n\}_{n\geqslant 1}$ *is a covering of the quasi-compact space* $|X|$*. Therefore* $V' \cup U_m = |X|$*, for some* $m \geqslant 1$*. In particular* $V_{m+1} \subseteq V'$*.*

Proof of Proposition 3.1.54. (i): By Corollary 3.1.51, the morphism $\coprod_{n\geqslant 1} U_n \to U$ is of universal !-descent. Since the class $E \supseteq$ qcqs is stable under disjoint unions and !-local on the source (c.f. Theorem 3.1.40) we conclude that $j \in E$.

(ii)(a): Let $j_n : U_n \to X$ be the inclusions. By Proposition 3.1.50, we know that $\operatorname{Cofib}(j_! j^! \to \mathrm{id}) \simeq \operatorname{colim}_n \operatorname{Cofib}(j_{n,*} j_n^! \to \mathrm{id})$. By Lemma 3.1.48, we know that

$$\operatorname*{colim}_n j_{n,*} j_n^! \simeq \operatorname*{colim}_n \operatorname{Fib}(\mathrm{id} \to k_{n,*} k_n^*) \simeq \operatorname{Fib}(\mathrm{id} \to \operatorname*{colim}_n k_{n,*} k_n^*), \tag{3.103}$$

where we used the property of stable ∞-categories. By using the property of stable ∞-categories this also implies that $\operatorname{Cofib}(\operatorname{colim}_n j_{n,*} j_n^! \to \mathrm{id}) \simeq \operatorname{colim}_n k_{n,*} k_n^*$, proving (ii)(a).

(ii)(b): Again let $j_n : U_n \to X$ be the inclusions. By descent we know that $\operatorname{Fib}(\mathrm{id} \to j_* j^*) \simeq \lim_n \operatorname{Fib}(\mathrm{id} \to j_{n,*} j_n^*)$. By Lemma 3.1.49 we know that

$$\lim_n j_{n,*} j_n^* \simeq \lim_n \operatorname{Cofib}(k_{n,*} k_n^! \to \mathrm{id}) \simeq \operatorname{Cofib}(\lim_n k_{n,*} k_n^! \to \mathrm{id}). \tag{3.104}$$

By using the property of stable ∞-categories this implies that $\operatorname{Fib}(\mathrm{id} \to \lim_n j_{n,*} j_n^*) \simeq \lim_n k_{n,*} k_n^!$, proving (ii)(b).

(iii): Let $k'_n : V'_n := V_n \cap U \to U$ and $j'_n : U_n \to U$ be the inclusions. By base-change, and since j^* is exact and colimit-preserving we deduce that $\operatorname{colim}_n k'_{n,*} k_n'^{,*} \simeq \operatorname{Cofib}(j^! \to j^*)$. Therefore, in order to prove the claim, it is enough to show that

$$\operatorname*{colim}_n k'_{n,*} k_n'^{,*} \simeq 0. \tag{3.105}$$

By descent, it is enough to check this after applying j_m^* for each $m \geqslant 1$. The functors j_m^* commute with colimits, and by base change, using that $U_m \cap V_{m+1} = \emptyset$, one has $j_m^* k'_{m+1,*} \simeq 0$. Therefore (3.105) is an equivalence and $j^! \simeq j^*$. □

3.1.8 Zariski-closed immersions

Definition 3.1.56. *Let* $f : X \to Y$ *be a morphism in* dRig*. We say that* f *is a* Zariski-closed immersion *if there exists a covering* $\{U_i \to Y\}_{i\in\mathcal{I}}$ *of* Y *by affinoid subspaces* $U_i = \mathrm{dSp}(A_i)$ *such that, for each* $i \in \mathcal{I}$*, the pullback* $X \times_Y U_i$ *is represented by an affinoid* $\mathrm{dSp}(B_i)$ *and the induced morphism* $A_i \to B_i$ *is surjective on* π_0*.*

Lemma 3.1.57. *(i) The class of Zariski-closed immersions is stable under composition and base-change.*

(ii) Every Zariski-closed immersion is quasi-compact.

Proof. (i): Omitted. (ii): Follows from Lemma 3.1.32(iii). ☐

Lemma 3.1.58. *Let $f : X \to Y$ be a Zariski-closed immersion in* dRig*. The image of $|X|$ under $|f|$ is a closed subset of $|Y|$.*

Proof. One reduces to the case when $Y = \mathrm{dSp}(A)$ is an affinoid and $X = \mathrm{dSp}(B) \to \mathrm{dSp}(A)$ is a morphism of affinoids such that $\pi_0 A \to \pi_0 B$ is surjective. By Remark 3.1.28 it is then sufficient to show that the image of $\mathrm{Spa}(\pi_0 B, (\pi_0 B)^\circ) \to \mathrm{Spa}(\pi_0 A, (\pi_0 A)^\circ)$ is closed. This is a consequence of [Hub96, 1.4.1]. ☐

3.1.9 Zariski-open immersions

Definition 3.1.59. *A morphism $f : X \to Y$ in* dRig *is called a* Zariski-open immersion *if there exists a Zariski-closed immersion $Z \to Y$ such that f is equivalent to the inclusion of the analytic subspace $U \hookrightarrow Y$ corresponding to $|Y| \setminus |Z| \subseteq |Y|$.*

Proposition 3.1.60. *Let $j : X \to Y$ be a Zariski-open immersion in* dRig*. Then:*

(i) The morphism $j : X \to Y$ belongs to the class E, i.e., it is !*-able.*

(ii) The canonical morphism $j^! \to j^*$ is an equivalence.

Proof. (i): By using that E is $*$*-local on the target* one reduces to the case when Y is an affinoid. Then the claim follows from Example 3.1.55 and Proposition 3.1.54(i).

(ii): We make the following temporary definitions. A Zariski-closed immersion $T \to S$ is called *basic* if $S = \mathrm{dSp}(A)$ and $T = \mathrm{dSp}(B)$ are both affinoid and $\pi_0 A \to \pi_0 B$ is surjective. A Zariski-open immersion $R \to S$ is called *basic* if S is affinoid and there exists a basic Zariski-closed immersion $T \to S$ such that R corresponds to the complement of $|T|$ in $|S|$. Now, we proceed in steps.

Step 1: When $X \to Y$ is a basic Zariski-open immersion, the statement follows from Example 3.1.55 and Proposition 3.1.54(iii).

Step 2: Now assume that Y is an analytic subspace of an affinoid subspace Y', and $X \to Y$ is induced by a basic Zariski-open immersion $X' \to Y'$. Choose a cover $\{U_i \to Y\}_{i \in \mathscr{I}}$ be a covering of Y by affinoid subspaces of Z. Let $\mathcal{I}$ be the family of finite nonempty subsets of $\mathscr{I}$. For each $I \in \mathcal{I}$ set $U_I := \bigcap_{i \in I} U_i$ and let $t_I : U_I \to Y$ be the inclusions. Each $X \cap U_I \to U_I$ is a basic Zariski-open immersion. By descent, base-change and using that $j^!$ commutes with limits, one has

$$j^! \simeq \lim_{I \in \mathcal{I}} j^! t_{I,*} t_I^* \simeq \lim_{I \in \mathcal{I}} t'_{I,*} j'^{,!} t_I^* \simeq \lim_{I \in \mathcal{I}} t'_{I,*} j'^{,*} t_I^* \simeq \lim_{I \in \mathcal{I}} t'_{I,*} t_I'^{,*} j^* \simeq j^*, \tag{3.106}$$

where we used Step 1.

Step 3: Now Y is arbitrary. By the Definition 3.1.56 of a Zariski-closed immersion, we may choose a cover $\{U_i \to Y\}_{i \in \mathscr{I}}$ be a covering of Y by affinoid subspaces such that each $U_i \cap Y \to U_i$ is a basic Zariski-open immersion. Let $\mathcal{I}$ be the family of finite nonempty subsets of $\mathscr{I}$. For each $I \in \mathcal{I}$ set $U_I := \bigcap_{i \in I} U_i$ and let $t_I : U_I \to Y$ be the inclusions. We proceed as in Step 2, using the result of Step 2, to conclude that $j^! \to j^*$ is an equivalence. ☐

3.1.10 Algebras of germs

For the sake of brevity let us introduce the following notations:

Definition 3.1.61. *We define* $\mathsf{dAlg} := \mathsf{CAlg}(D_{\geqslant 0}(\mathsf{CBorn}_K))$ *and* $\mathsf{dAff} := \mathsf{dAlg}^{\mathsf{op}}$. *The object of* dAff *corresponding to* $A \in \mathsf{dAlg}$ *is denoted by the formal expression* $\mathrm{dSp}(A)$. *For such we define* $\mathrm{QCoh}(\mathrm{dSp}(A)) := \mathrm{Mod}_A D(\mathsf{CBorn}_K)$. *We define* $\mathsf{PStk} := \mathrm{PSh}(\mathsf{dAff}) = \mathrm{Fun}(\mathsf{dAlg}, \infty\mathsf{Grpd})$.

Definition 3.1.62. *Let* $X = \mathrm{dSp}(A) \in \mathsf{dAfnd}$ *be a derived affinoid space and let* $i : \mathrm{dSp}(B) = Z \to X$ *be a Zariski-closed immersion defined by a morphism* $A \to B$ *which is surjective on* π_0. *The* algebra of germs along Z *is defined to be*

$$A_Z^{\dagger} := \underset{U \supseteq |Z|}{\mathrm{colim}} A_U \tag{3.107}$$

where the colimit is taken in $\mathsf{CAlg}(D_{\geqslant 0}(\mathsf{CBorn}_K))$ *and runs over all (affinoid) opens* $U \supseteq |Z|$. *We denote the corresponding object of* dAff *by* $(Z \subseteq X)^{\dagger}$.

Lemma 3.1.63. *With notations as in Definition 3.1.60. Let* $\iota : (Z \subseteq X)^{\dagger} \to X$ *be the canonical morphism. Then* ι *is a homotopy monomorphism.*

Proof. We must show that $A \to A_Z^{\dagger}$ is a homotopy epimorphism. Since the class of homotopy epimorphisms is stable under colimits, this follows immediately from the fact that each $A \to A_U$ is a homotopy epimorphism. □

Proposition 3.1.64. *With notations as in Definition 3.1.62. There is a natural equivalence of* ∞*-categories*

$$\mathrm{QCoh}((Z \subseteq X)^{\dagger}) \simeq \Gamma_Z \,\mathrm{QCoh}(X). \tag{3.108}$$

Proof. By Proposition 3.1.60 and Proposition 3.1.53 there is an equivalence between $\Gamma_Z \,\mathrm{QCoh}(X)$ and algebra objects in $\mathrm{QCoh}(X)$ over the idempotent algebra object

$$\mathrm{Cofib}(j_! j^! 1_X \to 1_X). \tag{3.109}$$

By Proposition 3.1.54(ii) and Example 3.1.55 we can identify this idempotent algebra object with $A_Z^{\dagger}$. Now the Proposition follows from the transitivity[19] property

$$\mathrm{Mod}_{A_Z^{\dagger}} \mathrm{Mod}_A \mathrm{QCoh}(*) \simeq \mathrm{Mod}_{A_Z^{\dagger}} \mathrm{QCoh}(*). \tag{3.110}$$

□

Corollary 3.1.65. *With notations as in Definition 3.1.60. Let* $\iota : (Z \subseteq X)^{\dagger} \to X$ *be the canonical morphism. Let* U *be the open subspace of* X *corresponding to the complement of* $|Z|$*, and let* $j : U \hookrightarrow X$ *be the inclusion. In the sequence*

$$\mathrm{QCoh}(U) \xrightarrow{j_!} \mathrm{QCoh}(X) \xrightarrow{\iota^*} \mathrm{QCoh}((Z \subseteq X)^{\dagger}) \tag{3.111}$$

the composite $\iota^* j_! \simeq 0$ *and the right adjoints* $j^!$ *and* ι_* *satisfy* $j^! j_! \simeq \mathrm{id}$ *and* $\iota^* \iota_* \simeq \mathrm{id}$. *Moreover the right adjoints* ι_* *and* $j^!$ *commute with colimits.*

Proof. Combine Proposition 3.1.53, Proposition 3.1.60 and Proposition 3.1.64. □

[19] For any symmetric monoidal ∞-category $(\mathscr{V}, \otimes)$ such that $\otimes$ is compatible with colimits separately in each variable, and any morphism $A \to B$ of commutative algebra objects in $\mathscr{V}$, there is an equivalence of ∞-categories

$$\mathrm{Mod}_B(\mathrm{Mod}_A \mathscr{V}) \simeq \mathrm{Mod}_B \mathscr{V};$$

on the left side here, we view B as a commutative algebra object in $\mathrm{Mod}_A \mathscr{V}$. This is proved by applying Barr–Beck–Lurie to the forgetful functor $\mathrm{Mod}_A \mathscr{V} \to \mathrm{Mod}_B \mathscr{V}$.

Definition 3.1.66. *We define the category* Pairs *as the full subcategory of* $\mathrm{Fun}(\Delta^1, \mathsf{dAfnd})$ *on objects* $Z = \mathrm{dSp}(B) \to \mathrm{dSp}(A) = X$ *induced by a morphism* $A \to B$ *which is surjective on* π_0.

The category Pairs has fiber products:

$$(Z \to X) \times_{(Z' \to X')} (Z'' \to X'') = (Z \times_{Z'} Z'' \to X \times_{X'} X''). \tag{3.112}$$

Lemma 3.1.67. *The functor* $\mathsf{Pairs} \to \mathsf{dAff} : (Z \to X) \mapsto (Z \subseteq X)^\dagger$ *preserves fiber products.*

Proof. We use notations as in (3.112). Say $X = \mathrm{dSp}(A)$ and $Z = \mathrm{dSp}(B)$, and similarly for Z', X', etc. Say $\pi_0 A \to \pi_0 B$ is defined by an ideal $I = (f_1, \ldots, f_r)$ and $\pi_0 A'' \to \pi_0 B''$ is defined by $I' = (f'_1, \ldots, f'_s)$. Then $\pi_0(A \otimes_{A'} A'') \to \pi_0(B \otimes_{B'} B'')$ is defined by

$$(f_1 \otimes 1, \ldots, f_r \otimes 1, 1 \otimes f''_1, \ldots, 1 \otimes f''_s). \tag{3.113}$$

Using the homeomorphisms $r : |\,\mathrm{Spa}(\pi_0 A, (\pi_0 A)^\circ)| \xrightarrow{\sim} |X|$, etc., we may define (rational) open subsets V_n of X as in Example 3.1.55:

$$r^{-1} V_n := \{x \in |\,\mathrm{Spa}(\pi_0 A, (\pi_0 A)^\circ)| : |f_i|_x \leqslant p^{-n} \text{ for all } 1 \leqslant i \leqslant k\}. \tag{3.114}$$

and similarly for V''_n. Then by cofinality of this system we find that the pushout of the algebras of germs is

$$\operatorname*{colim}_n A_{V_n} \widehat{\otimes}^{\mathbf{L}}_{A'} A''_{V''_n} = \operatorname*{colim}_n (A \widehat{\otimes}^{\mathbf{L}}_{A'} A'')_{V_n \times_{X'} V''_n} \tag{3.115}$$

and we note that $V_n \times_X V''_n$ is the open subset with

$$r^{-1}(V_n \times_X V''_n) = \{x : |f_i \otimes 1| \leqslant p^{-n}, |1 \otimes f''_j| \leqslant p^{-n} \text{ for all } i, j\}, \tag{3.116}$$

which is again a cofinal system of neighbourhoods of $Z \times_{Z'} Z''$. □

Lemma 3.1.68. *Let* $(Z \to X) \in \mathsf{Pairs}$. *Let affinoid opens* $\{U_i\}_{i=1}^n$ *of* X *be given such that* $|Z| \subseteq \bigcup_i U_i$. *Then:*

(i) Let $\mathcal{I}$ *be the family of finite nonempty subsets of* $\{1, \ldots, n\}$. *The natural morphism*

$$\mathrm{QCoh}^*((Z \subseteq X)^\dagger) \to \lim_{I \in \mathcal{I}} \mathrm{QCoh}^*((Z_I \subseteq X_I)^\dagger), \tag{3.117}$$

is an equivalence; here $Z_I := U_I \times_X Z$.

(ii) Let $Y := \coprod_i U_i$ *and* $S := \coprod_i Z_i \to Y$. *Then, the canonical morphism*

$$\mathrm{QCoh}((Z \subseteq X)^\dagger) \to \lim_{[m] \in \Delta^{\mathsf{op}}} \mathrm{QCoh}((S \subseteq Y)^{\dagger, m+1/(Z \subseteq X)^\dagger}) \tag{3.118}$$

is an equivalence.

Proof. Let us say $X = \mathrm{dSp}(A)$. According to [Kis99b, Lemma 2.3] and Lemma 3.1.9, the system of rational opens of X containing $|Z|$, is a cofinal system of open neighbourhoods of $|Z|$ in X. Hence we may find a rational open subset V with $|Z| \subseteq V \subseteq \bigcup_i U_i$. For such V one then has

$$A_V \xrightarrow{\sim} \lim_{(i_1, \ldots, i_k) \in \mathcal{I}} A_{V_{i_1}} \widehat{\otimes}^{\mathbf{L}}_A \ldots \widehat{\otimes}^{\mathbf{L}}_A A_{V_{i_k}}, \tag{3.119}$$

where $V_i = V \cap U_i$, c.f. the proof of Lemma 3.1.16 and especially (3.34). By compact generation (Proposition 2.1.49), filtered colimits commute with finite limits (Lemma 2.1.42), and hence we obtain

$$A_Z^\dagger \xrightarrow{\simeq} \lim_{(i_1,\dots,i_k)\in\mathcal{I}} A^\dagger_{Z_{i_1}} \widehat{\otimes}^{\mathbf{L}}_A \dots \widehat{\otimes}^{\mathbf{L}}_A A^\dagger_{Z_{i_k}}, \tag{3.120}$$

which by a standard argument implies (i). This also shows that $A_Z^\dagger \to \prod_i A^\dagger_{Z_i}$ is descendable, which gives (ii). □

3.1.11 Six-functor formalism for prestacks

In the next section we will work in a "bigger" six-functor formalism constructed as follows. We use notations as in Definition 3.1.61. We will apply the formalism of §2.3.2 in the following set-up (with notations as in that section):

- ⋆ We take $\mathscr{V} := D(\mathsf{CBorn}_K)$, so that $\mathscr{E} := \mathsf{CAlg}(D(\mathsf{CBorn}_K))^{\mathsf{op}}$ and we consider $\mathsf{dAff} = \mathsf{CAlg}(D_{\geqslant 0}(\mathsf{CBorn}_K))^{\mathsf{op}} \subseteq \mathscr{E}$. We take τ to be the trivial topology on dAff. We define $\mathsf{PStk} := \mathrm{Psh}(\mathsf{dAff})$.

It is clear that the assumptions of §3.2 are satisfied and hence we obtain:

Theorem 3.1.69. *The functor* QCoh *extends to a six-functor formalism*

$$\mathrm{QCoh} : \mathrm{Corr}(\mathsf{PStk}, \widetilde{E})^{\otimes} \to \mathsf{Pr}^{L,\otimes}_{\mathsf{st}} \tag{3.121}$$

such that the class $\widetilde{E} \supseteq \mathrm{rep}$ is stable under disjoint unions, $*$-local on the target, $!$-local on the source, is tame and satisfies $\widetilde{E} \subseteq \delta\widetilde{E}$. Further, every morphism $f \in \mathrm{rep}$ satisfies $f_! \simeq f_*$.

3.2 Stratifications and analytic $\mathcal{D}$-modules

3.2.1 Internal groupoid objects in an ∞-category

Definition 3.2.1. *Let $\mathscr{C}$ be an ∞-category admitting all fiber products.*

(i) A groupoid object *in $\mathscr{C}$ is a simplicial object $X \in s\mathscr{C} := \mathrm{Fun}(N(\Delta^{\mathsf{op}}), \mathscr{C})$ such that, for every $n \geqslant 0$ and for all subsets $S, S' \subseteq [n]$ with $S \cup S' = [n]$ and $|S \cap S'| = 1$, the diagram*

$$\begin{array}{ccc} X([n]) & \longrightarrow & X(S) \\ \downarrow & & \downarrow \\ X(S') & \longrightarrow & X(S\cap S') \end{array} \tag{3.122}$$

is Cartesian.

(ii) Assume that $\mathscr{C}$ admits geometric realizations. Let $X_\bullet$ be a groupoid object in $\mathscr{C}$. We obtain from $X_\bullet$ an augmented simplicial object of $\mathscr{C}$ by setting $X_{-1} := \mathrm{colim}_{[n]\in\Delta^{\mathsf{op}}} X_n$. We say that $X_\bullet$ is effective *if the canonical morphism $X_\bullet \to N(X_0 \to X_{-1})$ is an equivalence in $s\mathscr{C}$.*

Remark 3.2.2. *(i) Due to Definition 3.1.5(i), Definition 3.2.1(ii) is equivalent saying that $X_0 \to X_{-1}$ is an effective epimorphism.*

(ii) If $\mathscr{C}$ is an ∞-topos then every groupoid object in $\mathscr{C}$ is effective. Indeed, this is one of the Giraud-Rezk-Lurie axioms [Lur09b, Theorem 6.1.0.6].

Definition 3.2.3. *Let $\mathscr{C}$ be an ∞-category admitting all fiber products. We say that a morphism $X_\bullet \to Y_\bullet$ in $s\mathscr{C}$ is a* homotopy Kan fibration *if for all $n \geqslant 1$ and for all $0 \leqslant k \leqslant n$, the canonical morphism*

$$X(\Delta^n) \to X(\Lambda^n_k) \times_{Y(\Lambda^n_k)} Y(\Delta^n) \tag{3.123}$$

is an effective epimorphism.

Remark 3.2.4. *Let $\mathscr{C}$ be an ∞-category admitting all fiber products. It is immediate that the class of homotopy Kan fibrations in $s\mathscr{C}$ is stable under composition and base-change.*

Lemma 3.2.5. *Let $\mathscr{C}$ be an ∞-category admitting all finite limits and geometric realizations. Let $f : X_\bullet \to Y_\bullet$ be a morphism between groupoid objects of $\mathscr{C}$. Then, the following are equivalent.*

(i) f is a homotopy Kan fibration,

(ii) For $i = 0, 1$, the morphisms

$$X(\Delta^1) \to X(\Lambda^1_i) \times_{Y(\Lambda^1_i)} Y(\Delta^1), \tag{3.124}$$

are effective epimorphisms.

Proof. (i) $\implies$ (ii): This is trivial.

(ii) $\implies$ (i): Due to [Lur09b, Proposition 6.1.2.6(3)], for every $n \geqslant 2$ and $0 \leqslant k \leqslant n$ the canonical morphism $X(\Delta^n) \to X(\Lambda^n_k)$ is an equivalence. More precisely the result of *loc. cit.* says that the morphism $\mathscr{C}_{/X(\Delta^n)} \to \mathscr{C}_{/X(\Lambda^n_k)}$, induced by postcomposition, is an equivalence of ∞-categories. In particular, $[X(\Delta^n) \to X(\Lambda^n_k)]$ is terminal in $\mathscr{C}_{/X(\Lambda^n_k)}$, so we can deduce the claim from uniqueness of the terminal object. Obviously, the same is true for Y. Therefore, the condition (3.123) in the definition of a homotopy Kan fibration, is automatically satisfied for every $n \geqslant 2$. ☐

Proposition 3.2.6. *Let $\mathscr{X}$ be an ∞-topos. Let $X_\bullet \to Y_\bullet$ and $Z_\bullet \to Y_\bullet$ be morphisms in $s\mathscr{X}$. Assume that $X_\bullet \to Y_\bullet$ is a homotopy Kan fibration. Then, the canonical morphism*

$$|X_\bullet \times_{Y_\bullet} Z_\bullet| \to |X_\bullet| \times_{|Y_\bullet|} |Z_\bullet| \tag{3.125}$$

is an equivalence. Here we have abbreviated $|X_\bullet| := \operatorname{colim}_{[n] \in \Delta^{\mathsf{op}}} X_n$, etc.

Proof. The case when $\mathscr{X} = \infty\mathsf{Grpd}$ is proven in [MG15, Corollary 6.7]. The case when $\mathscr{X} = \mathrm{Psh}(\mathscr{D}, \infty\mathsf{Grpd})$ for some ∞-category $\mathscr{D}$, follows from the case when $\mathscr{C} = \infty\mathsf{Grpd}$, because limits and colimits in $\mathrm{Psh}(\mathscr{D}, \infty\mathsf{Grpd})$ are computed pointwise, c.f. [Lur09b, §5.1.2]. Now if $\mathscr{X}$ is an ∞-topos we may write $\mathscr{X}$ as a localization $L : \mathrm{Psh}(\mathscr{D}, \infty\mathsf{Grpd}) \leftrightarrows \mathscr{X} : i$ for some $\mathscr{D}$, where L is left exact. We recall that colimits in $\mathscr{X}$ are computed by first taking the colimit in $\mathrm{Psh}(\mathscr{D}, \infty\mathsf{Grpd})$ and then applying L, whence the claim follows. ☐

Let $\mathscr{X}$ be an ∞-topos. Motivated by [MG15], we define two classes of morphisms (which we call *weak equivalences* and *fibrations*) in $s\mathscr{X}$ as follows:

(W) The weak equivalences are precisely the morphisms in $s\mathscr{X}$ which are sent to equivalences under the geometric realization functor $|\cdot| : s\mathscr{X} \to \mathscr{X}$.

(F) The fibrations are the homotopy Kan fibrations.

For morphisms $X_\bullet \to Y_\bullet$ and $Z_\bullet \to Y_\bullet$ in $s\mathscr{X}$, Proposition 3.2.6 above gives us a strategy to calculate the fiber product $|X_\bullet| \times_{|Y_\bullet|} |Z_\bullet|$ as follows. One finds a factorization of the morphism $X_\bullet \to Y_\bullet$ as $X_\bullet \to X'_\bullet \to Y_\bullet$, where the first morphism is a weak equivalence and the second is a homotopy Kan fibration. Then, by Proposition 3.2.6, one has

$$|X_\bullet| \times_{|Y_\bullet|} |Z_\bullet| \simeq |X'_\bullet| \times_{|Y_\bullet|} |Z_\bullet| \simeq |X'_\bullet \times_{Y_\bullet} Z_\bullet|. \tag{3.126}$$

An example of this is the following. We may define an endofunctor $- \oplus [0]$ of the simplex category Δ, where $\oplus$ denotes the operation of *ordinal sum.* By precomposition, we then obtain an endofunctor $\mathrm{Dec}_0 : s\mathscr{X} \to s\mathscr{X}$, called *décalage.* For $X_\bullet \in s\mathscr{X}$, one has $(\mathrm{Dec}_0 X_\bullet)_n = X_{n+1}$.

Lemma 3.2.7. *Let $\mathscr{X}$ be an ∞-topos and let $X_\bullet$ be a groupoid object in $\mathscr{X}$. The morphism $X_0 \to X_\bullet$, where the former is viewed as a constant simplicial object, can be factored as*

$$X_0 \to \mathrm{Dec}_0 X_\bullet \to X_\bullet, \tag{3.127}$$

where the first morphism is a weak equivalence and the second is a homotopy Kan fibration. In particular, for any morphism $Y_\bullet \to X_\bullet$ in $s\mathscr{X}$ the fiber product $X_0 \times_{|X_\bullet|} |Y_\bullet|$ can be computed as $|(\mathrm{Dec}_0 X_\bullet) \times_{X_\bullet} Y_\bullet|$.

Proof. That $X_0 \to \mathrm{Dec}_0 X_\bullet$ is a weak equivalence is an immediate consequence of [Lur09b, Lemma 6.1.3.16]. To check that $\mathrm{Dec}_0 X_\bullet \to X_\bullet$ is a homotopy Kan fibration, by Lemma 3.2.5 it suffices to show that $X_2 \to X_1 \times_{X_0} X_1$ is an effective epimorphism, but this is even an equivalence (by assumption, $X_\bullet$ is a groupoid object). ☐

3.2.2 Formal theory of stratifications

Let $f : X \to Y$ be a morphism in dAfnd. Then, for every $n \geqslant 1$, the diagonal morphism $\Delta_{n,f} : X \to X^{n/Y}$ determines an object of the category Pairs (Definition 3.1.66). Hence we may consider the germ $(X \subseteq X^{n/Y})^\dagger$. In particular, the following definition makes sense.

Definition 3.2.8. *Let $f : X \to Y$ be a morphism in dAfnd.*

(i) The infinitesimal groupoid *of f, denoted $\mathrm{Inf}(X/Y)$, is the simplicial object of PStk with*

$$\mathrm{Inf}(X/Y)_n := (X \subseteq X^{n+1/Y})^\dagger. \tag{3.128}$$

(ii) The stratifying stack *of f, denoted $(X/Y)_{\mathrm{str}}$, is defined to be the object of PStk given by*

$$(X/Y)_{\mathrm{str}} := \operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} \mathrm{Inf}(X/Y)_n. \tag{3.129}$$

(iii) Let $X \in \mathsf{dAfnd}$. We define the infinitesimal groupoid of X *to be $\mathrm{Inf}(X) := \mathrm{Inf}(X/*)$ and the* stratifying stack *of X to be $X_{\mathrm{str}} := (X/*)_{\mathrm{str}}$.*

Remark 3.2.9. *(i) Let $f : X \to Y$ be a morphism in dAfnd. Then $\mathrm{Inf}(X/Y)$ is a groupoid object of dAff in the sense of Definition 3.2.1. This follows from Lemma 3.1.67.*

(ii) For a fixed object $Y \in \mathsf{dAfnd}$, these constructions define functors

$$\mathrm{Inf}(-/Y) : \mathsf{dAfnd}_{/Y} \to s\mathsf{dAfnd}_{/Y} \quad \textit{and} \quad (-/Y)_{\mathrm{str}} : \mathsf{dAfnd}_{/Y} \to \mathsf{PStk}_{/Y}.$$

(iii) Let $k : X \to Z$, $h : Y \to Z$ *be morphisms of* dAfnd *and let* $f : X \to Y$ *be a morphism over* Z*. Due to Lemma 3.2.5, the following are equivalent:*

(a) $\mathrm{Inf}(X/Z) \to \mathrm{Inf}(Y/Z)$ *is a homotopy Kan fibration,*

(b) The morphism $(X \subseteq X \times_Z X)^\dagger \to (X \subseteq X \times_Z Y)^\dagger$ *induced by* (id, f)*, is an effective epimorphism in* PStk*. Here the latter morphism is the graph of* f*.*

Example 3.2.10. *(i) Let* $X \in$ dAfnd *and let* $X \to \mathrm{dSp}(K)$ *be the structure morphism. Then, by Remark 3.2.9(iii), the morphism* $\mathrm{Inf}(X) \to \mathrm{Inf}(\mathrm{dSp}(K)) \simeq \mathrm{dSp}(K)$*, is a homotopy Kan fibration.*

(ii) Let $X \in$ dAfnd *and let* $f : U \to X$ *be the inclusion of an affinoid open subspace. Then* $(U \subseteq U \times U)^\dagger \to (U \subseteq U \times X)^\dagger$ *is an equivalence in* dAfnd *because* $U \times U$ *is a neighbourhood of* U *(embedded via the graph) in* $U \times X$*. Hence, by Remark 3.2.9(iii),* $\mathrm{Inf}(U) \to \mathrm{Inf}(X)$ *is a homotopy Kan fibration.*

(iii) If $U \to X$ *is any morphism in* dAfnd *such that* $U \to U \times_X U$ *is an open immersion, then the canonical morphism* $U \to (U/X)_{\mathrm{str}}$ *is an equivalence. This is because, for each* $n \geqslant 1$*, there is an equivalence* $U \simeq (U \subseteq U^{n+1/X})^\dagger$*, by construction of the germ. In particular, this includes the case of (ii) above.*

Lemma 3.2.11. *Let* $f : X \to Y$ *and* $g : Y' \to Y$ *be morphisms in* dAfnd*. Set* $X' := X \times_Y Y'$*. Assume that the induced morphism* $\mathrm{Inf}(X) \to \mathrm{Inf}(Y)$ *is a homotopy Kan fibration. Then:*

(i) The canonical morphism $(X/Y)_{\mathrm{str}} \to X_{\mathrm{str}} \times_{Y_{\mathrm{str}}} Y$ *is an equivalence.*

(ii) The canonical morphism $X'_{\mathrm{str}} \to X_{\mathrm{str}} \times_{Y_{\mathrm{str}}} Y'_{\mathrm{str}}$ *is an equivalence.*

(iii) The morphism $\mathrm{Inf}(X') \to \mathrm{Inf}(Y')$ *is also a homotopy Kan fibration and the canonical morphism* $(X'/Y')_{\mathrm{str}} \to (X/Y)_{\mathrm{str}} \times_Y Y'$ *is an equivalence.*

(iv) Assume that there are morphisms $h : Y \to Z$ *and* $k : X \to Z$ *in* dAfnd *making* f *into a morphism over* Z*. Then, the induced morphism* $\mathrm{Inf}(X/Z) \to \mathrm{Inf}(Y/Z)$ *is a homotopy Kan fibration and the natural morphism* $(X/Y)_{\mathrm{str}} \to (X/Z)_{\mathrm{str}} \times_{(Y/Z)_{\mathrm{str}}} Y$ *is an equivalence.*

Proof. (i): We regard Y as a constant simplicial object, equipped with a morphism $Y \to \mathrm{Inf}(Y)$ induced by the diagonal morphisms

$$Y \to (Y \subseteq Y^{n+1})^\dagger = \mathrm{Inf}(Y)_n. \tag{3.130}$$

In particular, we calculate

$$Y \times_{\mathrm{Inf}(Y)_n} \mathrm{Inf}(X)_n \simeq (X \subseteq X^{n+1/Y})^\dagger = \mathrm{Inf}(X/Y)_n. \tag{3.131}$$

Hence, by Proposition 3.2.6, we see that $(X/Y)_{\mathrm{str}} \xrightarrow{\sim} X_{\mathrm{str}} \times_{Y_{\mathrm{str}}} Y$.

(ii): We calculate

$$\mathrm{Inf}(X)_n \times_{\mathrm{Inf}(Y)_n} \mathrm{Inf}(Y')_n \simeq (X \times_Y Y' \subseteq (X \times_Y Y')^{n+1})^\dagger \simeq \mathrm{Inf}(X')_n, \tag{3.132}$$

and so by Proposition 3.2.6 again we conclude that $X'_{\mathrm{str}} \xrightarrow{\sim} X_{\mathrm{str}} \times_{Y_{\mathrm{str}}} Y'_{\mathrm{str}}$.

(iii): By the calculation (3.132), and the fact that homotopy Kan fibrations are stable under base-change, we see that $\mathrm{Inf}(X') \to \mathrm{Inf}(Y')$ is a homotopy Kan fibration. We calculate

$$\begin{aligned} Y' \times_Y \mathrm{Inf}(X/Y)_n &\simeq Y' \times_Y (X \subseteq X^{n+1/Y})^{\dagger} \\ &\simeq (X' \subseteq X'^{,n+1/Y'})^{\dagger} \\ &= \mathrm{Inf}(X'/Y')_n. \end{aligned} \tag{3.133}$$

The morphism $\mathrm{Inf}(X/Y) \to Y$ is always a homotopy Kan fibration, by Lemma 3.2.5. Therefore by Proposition 3.2.6 we conclude that $(X'/Y')_{\mathrm{str}} \xrightarrow{\sim} (X/Y)_{\mathrm{str}} \times_Y Y'$.

(iv): We note that $\mathrm{Inf}(X/Z) \to \mathrm{Inf}(Y/Z)$ is the pullback of $\mathrm{Inf}(X) \to \mathrm{Inf}(Y)$ along $\mathrm{Inf}(Y/Z) \to \mathrm{Inf}(Y)$, and is therefore a homotopy Kan fibration. For each $n \geqslant 0$ the following square is also Cartesian:

$$\begin{array}{ccc} (X \subseteq X^{n+1/Y})^{\dagger} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ (X \subseteq X^{n+1/Z})^{\dagger} & \longrightarrow & (Y \subseteq Y^{n+1/Z})^{\dagger} \end{array} \tag{3.134}$$

so that $\mathrm{Inf}(X/Y) \xrightarrow{\sim} \mathrm{Inf}(X/Z) \times_{\mathrm{Inf}(Y/Z)} Y$. Therefore, by Proposition 3.2.6 we conclude that $(X/Y)_{\mathrm{str}} \xrightarrow{\sim} (X/Z)_{\mathrm{str}} \times_{(Y/Z)_{\mathrm{str}}} Y$. □

Lemma 3.2.12. *Let $f : X \to Y$ be a morphism in* dAfnd*. The canonical morphism $X \to (X/Y)_{\mathrm{str}}$ belongs to the class* rep *of representable morphisms in* PStk*.*

Proof. Let us take $Z \in \mathsf{dAff} \subseteq \mathsf{PStk}$ with a morphism $Z \to (X/Y)_{\mathrm{str}}$; we need to show that the pullback $Z \times_{(X/Y)_{\mathrm{str}}} X$ is representable. In any category of presheaves, all representable objects are projective. In particular the map $Z \to (X/Y)_{\mathrm{str}}$ has the right lifting property against effective epimorphisms, so there exists a lift $Z \to X$, up to homotopy. Using this lift, and the associativity of fiber products, we deduce that

$$Z \times_{(X/Y)_{\mathrm{str}}} X \simeq Z \times_X X \times_{(X/Y)_{\mathrm{str}}} X \simeq Z \times_X (X \subseteq X \times_Y X)^{\dagger}, \tag{3.135}$$

where we used that groupoid objects are effective. Since dAff is closed under fiber products, we deduce that $X \to (X/Y)_{\mathrm{str}}$ belongs to rep. □

Definition 3.2.13. *A morphism $f : X \to Y$ in* dAfnd *is called* good *if:*

- ⋆ *The morphism $\mathrm{Inf}(X) \to \mathrm{Inf}(Y)$ is a homotopy Kan fibration,*
- ⋆ *The morphism $X \to (X/Y)_{\mathrm{str}}$ is of universal !-descent, with respect to the six-functor formalism on* PStk *(Theorem 3.1.69). This condition makes sense by Lemma 3.2.12.*

Lemma 3.2.14. *(i) The class of good morphisms in* dAfnd *is stable under base-change and composition. The functor $(-)_{\mathrm{str}} : \mathsf{dAfnd} \to \mathsf{PStk}$ preserves finite products, and pullbacks of edges in the class* good*.*

(ii) If $f : X \to Y$ is good then $f_{\mathrm{str}} : X_{\mathrm{str}} \to Y_{\mathrm{str}}$ belongs to the class E of !-able morphisms in the six-functor formalism on PStk *(Theorem 3.1.69).*

Before proving the Lemma, we make note of an immediate Corollary.

Corollary 3.2.15. *The functor $(-)_{\mathrm{str}}$ induces a symmetric-monoidal functor*

$$(-)_{\mathrm{str}} : \mathrm{Corr}(\mathsf{dAfnd}, \mathrm{good}) \to \mathrm{Corr}(\mathsf{PStk}, \widetilde{E}). \tag{3.136}$$

where $\widetilde{E}$ is the class of edges in PStk *coming from Theorem 3.1.69.*

Proof of Lemma 3.2.14. (i): All of these properties follow from Lemma 3.2.11.

(ii): Since the class $\widetilde{E}$ is $*$-*local on the target*, it suffices to check that $f_{\mathrm{str}} \in \widetilde{E}$ after pullback along a morphism $Z \to Y_{\mathrm{str}}$ from a representable object of PStk. Again, since representable objects are projective, this lifts to a morphism $Z \to Y$, up to homotopy. Using this morphism and the associativity of fiber products we see that

$$Z \times_{Y_{\mathrm{str}}} X_{\mathrm{str}} \simeq Z \times_Y Y \times_{Y_{\mathrm{str}}} X_{\mathrm{str}} \simeq Z \times_Y (X/Y)_{\mathrm{str}}, \tag{3.137}$$

where we used Lemma 3.2.11(i). Therefore, since the class $\widetilde{E}$ is stable under base-change, it suffices to show that $(X/Y)_{\mathrm{str}} \to Y$ belongs to $\widetilde{E}$. The morphism $X \to Y$ factors as $X \to (X/Y)_{\mathrm{str}} \to Y$, where the first morphism is of universal !-descent by assumption. Therefore, since the class $\widetilde{E}$ is !-*local on the source*, we deduce that $(X/Y)_{\mathrm{str}} \to Y \in \widetilde{E}$. □

By Corollary 3.2.15 we now have a symmetric-monoidal functor

$$(-)_{\mathrm{str}} : \mathrm{Corr}(\mathsf{dAfnd}, \mathrm{good}) \to \mathrm{Corr}(\mathsf{PStk}, \widetilde{E}). \tag{3.138}$$

By post-composing $(-)_{\mathrm{str}}$ with the six-functor formalism QCoh on $(\mathsf{PStk}, \widetilde{E})$, we obtain a six-functor formalism

$$\mathrm{Strat} := \mathrm{QCoh} \circ (-)_{\mathrm{str}} : \mathrm{Corr}(\mathsf{dAfnd}, \mathrm{good}) \to \mathsf{CAlg}(\mathsf{Pr}^L_{\mathsf{st}}). \tag{3.139}$$

We would like to extend this six-functor formalism to all objects of dRig and also to a much larger class than just the good morphisms. Unfortunately, the class of good morphisms is not closed under the formation of diagonals, so we cannot apply the extension formalism of §2.3.1 and we have to proceed in a more ad-hoc manner.

Definition 3.2.16. *We define E_{str} to be the class of morphisms $f : X \to Y$ in* dRig *which are representable in* good*. That is, for any morphism $Y' \to Y$ from an object of* dAfnd*, the pullback $f' : X' \to Y'$ is a morphism between objects of* dAfnd *which belongs to the class* good*.*

Therefore, by [Man22, Proposition A.5.16] again, the six-functor formalism of (3.139) extends to a six-functor formalism

$$\mathrm{Strat} : \mathrm{Corr}(\mathsf{dRig}, E_{\mathrm{str}}) \to \mathsf{CAlg}(\mathsf{Pr}^L_{\mathsf{st}}), \tag{3.140}$$

uniquely such that

$$\mathrm{Strat}^*(X) \xrightarrow{\sim} \lim_{Y \in \mathsf{dAfnd}^{\mathsf{op}}_{/X}} \mathrm{Strat}^*(Y), \tag{3.141}$$

for all $X \in \mathsf{dRig}$. In particular QCoh^* is the left Kan extension of its restriction to dAfnd. One could further iterate the extension principles of [Man22, §A.5], although we do not do this here for the reason stated above. Let $f : X \to Y$ be a morphism in dRig. We will denote the six operations of the six-functor formalism Strat by

$$(f^*_{\mathrm{str}}, f_{\mathrm{str},*}, f_{\mathrm{str},!}, f^!_{\mathrm{str}}, \widehat{\otimes}_{X_{\mathrm{str}}}, \underline{\mathrm{Hom}}_{X_{\mathrm{str}}}), \tag{3.142}$$

where, of course, the functors $f_{\mathrm{str},!}$ and $f^!_{\mathrm{str}}$ are only defined when $f \in E_{\mathrm{str}}$.

3.2.3 Descent and Kashiwara's equivalence

Let $f : X \to Y$ be any morphism in dAfnd. The equivalence

$$\mathrm{QCoh}((X/Y)_{\mathrm{str}}) \xrightarrow{\sim} \lim_{[n]\in\Delta} \mathrm{QCoh}^*((X \subseteq X^{n+1/Y})^\dagger) \tag{3.143}$$

follows automatically from the fact that QCoh^* is a limit-preserving functor on $\mathsf{PStk}^{\mathsf{op}}$. Using this equivalence, it is quite easy to show the following.

Lemma 3.2.17. *Let $X \in \mathsf{dAfnd}$. Let $\{U_i \to X\}_{i=1}^n$ be a finite cover of X by affinoid subspaces.*

(i) Let $\mathcal{I}$ be the family of finite nonempty subsets of $\{1, \dots, n\}$ and for each $I \in \mathcal{I}$ set $U_I := \bigcap_{i\in I} U_i$. Then, the canonical morphism

$$\mathrm{Strat}^*(X) \to \lim_{I\in\mathcal{I}} \mathrm{Strat}^*(U_I) \tag{3.144}$$

is an equivalence.

(ii) Set $Y := \coprod_{i=1}^n U_i \to X$. Then, the canonical morphism

$$\mathrm{Strat}^*(X) \to \lim_{[m]\in\Delta} \mathrm{Strat}^*(Y^{m+1/X}) \tag{3.145}$$

is an equivalence.

Proof. (i): Let $t_I : U_I \to X$ be the inclusions. By using the presentation

$$\mathrm{Strat}(X) \xrightarrow{\sim} \lim_{[m]\in\Delta} \mathrm{QCoh}^*((X \subseteq X^{n+1})^\dagger), \tag{3.146}$$

it is sufficient to show that, for each $m \geqslant 0$, the canonical morphism

$$\mathrm{QCoh}^*((X \subseteq X^{m+1})^\dagger) \to \lim_{I\in\mathcal{I}} \mathrm{QCoh}^*((U_I \subseteq U_I^{m+1})^\dagger) \tag{3.147}$$

is an equivalence. However, this follows from Lemma 3.1.68(i), since the diagonally embedded copy of $|X|$ in X^{m+1} is contained in $\bigcup_{i=1}^n U_i^{\times m+1}$.

(ii): This is quite similar to the proof of (i), using Lemma 3.1.68(ii) instead of Lemma 3.1.68(i). ☐

Corollary 3.2.18. *The prestack* $\mathrm{Strat}^* : \mathsf{dRig} \to \mathsf{CAlg}(\mathsf{Pr}^L_{\mathsf{st}})$ *is a sheaf in the analytic topology.*

Proof. Since Strat^* is right Kan extended from $\mathsf{dAfnd}^{\mathsf{op}}$, the combination of Lemma 3.2.17 and [Man22, Proposition A.3.11] gives the Corollary. ☐

Let $X \in \mathsf{dRig}$ and let $S \subseteq |X|$ be a closed subset of the underlying topological space. Let $j : U \to X$ be the inclusion of the open analytic subspace corresponding to to complement of S. We define

$$\Gamma_S \,\mathrm{Strat}(X) \subseteq \mathrm{Strat}(X) \tag{3.148}$$

to be the full subcategory spanned by objects M such that $j^*_{\mathrm{str}} M \simeq 0$.

Proposition 3.2.19. *Let $i : Z \to X$ be a Zariski-closed immersion in* dAfnd *which is induced by a morphism of algebras which is surjective on π_0. Assume that i admits a retraction $r : X \to Z$. Then:*

(i) There is a canonical equivalence

$$(Z \subseteq X)^\dagger \simeq Z_{\mathrm{str}} \times_{X_{\mathrm{str}}} X. \tag{3.149}$$

in PStk*.*

(ii) In the Cartesian square

$$\begin{array}{ccc} (Z \subseteq X)^\dagger & \xrightarrow{\ \iota\ } & X \\ \downarrow{\scriptstyle q} & \lrcorner & \downarrow{\scriptstyle p} \\ Z_{\mathrm{str}} & \xrightarrow{\ i_{\mathrm{str}}\ } & X_{\mathrm{str}} \end{array} \tag{3.150}$$

coming from (i), the Beck-Chevalley morphism

$$p^* i_{\mathrm{str},*} \to \iota_* q^* \tag{3.151}$$

is an equivalence of functors from $\mathrm{Strat}(Z) = \mathrm{QCoh}(Z_{\mathrm{str}})$ *to* $\mathrm{QCoh}(X)$*.*

(iii) The pair $(i^*_{\mathrm{str}}, i_{\mathrm{str},*})$ induces an equivalence $\mathrm{Strat}(Z) \simeq \Gamma_{|Z|}\,\mathrm{Strat}(X)$*.*

Proof. (i): In order to calculate $X \times_{X_{\mathrm{str}}} Z_{\mathrm{str}}$, we use décalage, c.f. Lemma 3.2.7. We note that $(\mathrm{Dec}_0\,\mathrm{Inf}(X)) \times_{\mathrm{Inf}(X)} \mathrm{Inf}(Z)$ is given by the simplicial object $(Z \subseteq X \times Z^{\bullet+1})^\dagger$, where Z is embedded via the morphism $(i, \Delta_{n+1}) : Z \to X \times Z^{\times n+1}$. We recognise this simplicial object as the Čech nerve of $(X \subseteq X \times Z)^\dagger \to (Z \subseteq X)^\dagger$. This is a split epimorphism: the splitting is induced by $(\mathrm{id}, r) : X \to X \times Z$. Therefore, we conclude that the augmented simplicial object $(Z \subseteq X \times Z^{\bullet+1})^\dagger \to (Z \subseteq X)^\dagger$ is split, as it is the nerve of a split epimorphism. By Lemma 3.2.7, this proves that $(Z \subseteq X)^\dagger \simeq Z_{\mathrm{str}} \times_{X_{\mathrm{str}}} X$.

(ii): The functor QCoh^*, by its construction, satisfies descent along $X \to X_{\mathrm{str}}$. By (i), the pullback of each covering map $X^{n+1/X_{\mathrm{str}}} \simeq (X \subseteq X^{n+1})^\dagger \to X_{\mathrm{str}}$ along $Z_{\mathrm{str}} \to X_{\mathrm{str}}$ is given by $\iota_n : (Z \subseteq X^{n+1})^\dagger \to (X \subseteq X^{n+1})^\dagger$. Each pushforward $\iota_{n,*}$ is compatible with base-change and therefore, for each Cartesian section

$$(M_n)_{n\in\Delta} \in \lim_{[n]\in\Delta} \mathrm{QCoh}((Z \subseteq X^{n+1})^\dagger) \simeq \mathrm{QCoh}(Z_{\mathrm{str}}) \tag{3.152}$$

then

$$(\iota_{n,*} M_n)_{n\in\Delta} \in \lim_{[n]\in\Delta} \mathrm{QCoh}((X \subseteq X^{n+1})^\dagger) \simeq \mathrm{QCoh}(X_{\mathrm{str}}) \tag{3.153}$$

is also a Cartesian section. By the equivalence of categories implicit in descent, this implies that $p^* i_{\mathrm{str},*} \simeq \iota_* q^*$.

(iii): Let $j : U \to X$ be the inclusion of the analytic subspace corresponding to $|X| \setminus |Z| \subseteq |X|$. Consider the following diagram in PStk, in which both squares are Cartesian (for the left square this is (i) and for the right square this is Example 3.2.10):

$$\begin{array}{ccccc} (Z \subseteq X)^\dagger & \xrightarrow{\ \iota\ } & X & \xleftarrow{\ j\ } & U \\ {\scriptstyle q}\downarrow & \lrcorner & {\scriptstyle p}\downarrow & \llcorner & \downarrow{\scriptstyle r} \\ Z_{\mathrm{str}} & \xrightarrow{\ i_{\mathrm{str}}\ } & X_{\mathrm{str}} & \xleftarrow{\ j_{\mathrm{str}}\ } & U_{\mathrm{str}} \end{array} \tag{3.154}$$

We make two claims: (a) that the counit morphism $i_{\mathrm{str}}^* i_{\mathrm{str},*} \to \mathrm{id}$ is an equivalence, and (b) that $j_{\mathrm{str}}^* M \simeq 0$ if and only if the unit morphism $M \to i_{\mathrm{str},*} i_{\mathrm{str}}^* M$ is an equivalence.

We note that each of the functors q^*, p^*, r^* is conservative (each of the morphisms p, q, r is of $*$-descent, because they are effective epimorphisms). In particular, it suffices to check that (a) is an equivalence after applying q^*. By commutativity of (3.154), and the base-change of part (ii), one has $q^* i_{\mathrm{str}}^* i_{\mathrm{str},*} \simeq \iota^* \iota_* q^*$, and $\iota^* \iota_* \to \mathrm{id}$ is an equivalence, because $(Z \subseteq X)^\dagger \to X$ is a homotopy monomorphism.

For (b), we have the following chain of equivalences, for $M \in \mathrm{QCoh}(X_{\mathrm{str}})$:

$$\begin{aligned}
j_{\mathrm{str}}^* M \simeq 0 &\iff r^* j_{\mathrm{str}}^* M \simeq 0 && \text{by conservativity of } r^* \\
&\iff j^* p^* M \simeq 0 && \text{by commutativity of (3.154)} \\
&\iff \iota_* \iota^* p^* M \xrightarrow{\sim} p^* M && \text{by Corollary 3.1.65} \\
&\iff p^* i_{\mathrm{str},*} i_{\mathrm{str}}^* M \xrightarrow{\sim} p^* M && \text{by part (ii) and commutativity of (3.154)} \\
&\iff i_{\mathrm{str},*} i_{\mathrm{str}}^* M \xrightarrow{\sim} M && \text{by conservativity of } p^*.
\end{aligned}$$

It then follows from the claims (a) and (b) that $(i_{\mathrm{str}}^*, i_{\mathrm{str},*})$ induces an equivalence $\mathrm{Strat}(Z) \simeq \Gamma_{|Z|} \mathrm{Strat}(X)$. Indeed, for inclusions of coreflective subcategories, the essential image of the fully-faithful right adjoint is precisely those objects for which the unit morphism is an equivalence. □

Lemma 3.2.20. *Let $X \in \mathsf{dRig}$ with $S \subseteq |X|$ a closed subset of the underlying topological space. Let open subsets $\{U_i\}_{i \in \mathscr{I}}$ of $|X|$ be given such that $S \subseteq \bigcup_{i \in \mathscr{I}} U_i$. Let $\mathcal{I}$ be the family of finite nonempty subsets of $\mathscr{I}$. Then, the natural morphism*

$$\Gamma_S \mathrm{Strat}(X) \to \lim_{I \in \mathcal{I}} \Gamma_{S_I} \mathrm{Strat}(U_I) \tag{3.155}$$

induced by the collection of upper-star functors, is an equivalence.

Proof. This follows quite straightforwardly from the fact that Strat^* is a sheaf in the analytic topology, c.f. Lemma 3.2.17. □

Definition 3.2.21. *Let $X \in \mathsf{dRig}$. A Zariski-closed immersion $Z \to X$ is called* stratifying *if it locally admits a retraction. That is, there exists affinoid subspaces $\{U_i\}_{i \in \mathscr{I}}$ of X such that $|Z| \subseteq \bigcup_{i \in \mathscr{I}} U_i$, and for all $i \in \mathscr{I}$, the morphism $Z \times_X U_i \to U_i$ admits a retraction.*

Remark 3.2.22. *(i) I chose the name* stratifying *because a similar kind of closed immersions appears in the definition of the stratifying site in algebraic geometry [Gro68, §4.2].*

(ii) In analytic geometry, it seems quite plausible that a "Kashiwara's equivalence" could hold for a much larger class of immersions than just Zariski-closed immersions.

(iii) If $i : Z \to X$ is a closed immersion between smooth classical affinoid rigid spaces, then i is stratifying. This follows from a result of Kiehl [Kie67, Theorem 1.19], see also [BLR95, Proposition 2.11].

Theorem 3.2.23 (Kashiwara's equivalence)**.** *Let $i : Z \to X$ be a stratifying Zariski-closed immersion in dRig. Then, the pair $(i_{\mathrm{str}}^*, i_{\mathrm{str},*})$ induces an equivalence*

$$\mathrm{Strat}(Z) \simeq \Gamma_{|Z|} \mathrm{Strat}(X). \tag{3.156}$$

Proof. We proceed in steps.

Step 1: If $X \in \mathsf{dRig}$ is an affinoid and $i : Z \to X$ admits a retraction, this follows from Proposition 3.2.19.

Step 2: Now suppose that X is an analytic subspace of an affinoid space X' and $i : Z \to X$ is a Zariski-closed immersion which admits a retraction. Choose a covering $\{U_i \to X\}_{i \in \mathscr{I}}$ of X by affinoid subspaces of X'. Let $\mathcal{I}$ be the family of finite nonempty subsets of $\mathscr{I}$ and for each $I \in \mathcal{I}$ set $U_I := \bigcup_{i \in I} U_i$. Each U_I is an affinoid and each $Z_I := Z \times_X U_I \to U_I$ admits a retraction. We have a commutative square

$$\begin{array}{ccc} \Gamma_{|Z|}\,\mathrm{Strat}(X) & \longrightarrow & \lim_{I \in \mathcal{I}} \Gamma_{|Z_I|}\,\mathrm{Strat}(U_I) \\ \downarrow & & \downarrow \\ \mathrm{Strat}(Z) & \longrightarrow & \lim_{I \in \mathcal{I}} \mathrm{Strat}(Z_I) \end{array} \tag{3.157}$$

in which the horizontal arrows are equivalences by Lemma 3.2.17 and Lemma 3.2.20, and the right vertical arrow is an equivalence by Step 1. Therefore, the left vertical arrow is an equivalence.

Step 3: Now X is arbitrary. Choose affinoid subspaces $\{U_i\}_{i \in \mathcal{I}}$ of X such that $|Z| \subseteq \bigcup_{i \in \mathscr{I}} U_i$ and each $Z_I := Z \times_X U_i \to U_I$ admits a retraction. Each U_I is an analytic subspace of an affinoid space, and each $Z_I \to U_I$ admits a retraction. Therefore, we may argue as in Step 2, using the result of Step 2, to conclude the proof. □

3.2.4 The monad of differential operators and the comonad of jets

Definition 3.2.24. *Let $f : X \to Y$ be a morphism in dAfnd and let $p_{X/Y} : X \to (X/Y)_{\mathrm{str}}$ be the canonical morphism.*

(i) We define the comonad of jets *of f to be the comonad*

$$\mathcal{J}^{\infty}_{X/Y} := p^*_{X/Y} p_{X/Y,*} \tag{3.158}$$

acting on QCoh(X)*. We abbreviate $\mathcal{J}^{\infty}_{X/*}$ to $\mathcal{J}^{\infty}_X$.*

(ii) We define the monad of differential operators *of f to be the monad*

$$\mathcal{D}^{\infty}_{X/Y} := p^!_{X/Y} p_{X/Y,!} \tag{3.159}$$

acting on QCoh(X)*. We abbreviate $\mathcal{D}^{\infty}_{X/*}$ to $\mathcal{D}^{\infty}_X$.*

Lemma 3.2.25. *Let $f : X \to Y$ be a morphism in dAfnd. Let $p_{X/Y} : X \to (X/Y)_{\mathrm{str}}$ be the canonical morphism. Then, there is a canonical equivalence $p_{X/Y,!} \xrightarrow{\sim} p_{X/Y,*}$.*

Proof. This is an immediate consequence of Lemma 3.2.12, because the morphism $p_{X/Y}$ is even representable in PStk, and all such morphisms satisfy $p_{X/Y,!} \simeq p_{X/Y,*}$, c.f. Theorem 3.1.69. □

Remark 3.2.26. *(i) Let $f : X = \mathrm{dSp}(A) \to \mathrm{dSp}(B) = Y$ be a morphism between affinoids. Due to Lemma 3.2.25 and base-change, the underlying endofunctors of $\mathcal{J}^{\infty}_{X/Y}$ and $\mathcal{D}^{\infty}_{X/Y}$ can be described as*

$$\begin{aligned} \mathcal{J}^{\infty}_{X/Y} &\simeq \tilde{\pi}_{1,*}\tilde{\pi}_2^* \simeq (A \widehat{\otimes}^{\mathbf{L}}_B A)^{\dagger}_{\Delta} \otimes^{\mathbf{L}}_A - \\ \mathcal{D}^{\infty}_{X/Y} &\simeq \tilde{\pi}_{2,!}\tilde{\pi}_1^! \simeq R\underline{\mathrm{Hom}}_A((A \widehat{\otimes}^{\mathbf{L}}_B A)^{\dagger}_{\Delta}, -). \end{aligned} \tag{3.160}$$

Here $\tilde{\pi}_1, \tilde{\pi}_2 : (X \subseteq X \times_Y X)^{\dagger} \to X$ are the two projections.

(ii) As a formal consequence of Lemma 3.2.25, we see that the monad $\mathcal{D}^{\infty}_{X/Y}$ is right adjoint to the comonad $\mathcal{J}^{\infty}_{X/Y}$. In particular, their Kleisli categories are equivalent. Morphisms in either Kleisli category can be interpreted as infinite-order differential operators.

Lemma 3.2.27. *Let $f : X \to Y$ be a morphism in dAfnd and let $p_{X/Y} : X \to (X/Y)_{\mathrm{str}}$ be the canonical morphism.*

(i) The adjunction $p^*_{X/Y} \dashv p_{X/Y,*}$ is comonadic. That is, the comparison functor induced by $p^*_{X/Y}$ gives an equivalence of categories

$$\mathrm{QCoh}((X/Y)_{\mathrm{str}}) \simeq \mathrm{Comod}_{\mathcal{J}^{\infty}_{X/Y}} \mathrm{QCoh}(X). \tag{3.161}$$

(ii) Assume that $X \to (X/Y)_{\mathrm{str}}$ is of !-descent. Then the adjunction $p_{X/Y,!} \dashv p^!_{X/Y}$ is monadic. That is, the comparison functor induced by $p^!_{X/Y}$ gives an equivalence of categories

$$\mathrm{QCoh}((X/Y)_{\mathrm{str}}) \simeq \mathrm{Mod}_{\mathcal{D}^{\infty}_{X/Y}} \mathrm{QCoh}(X). \tag{3.162}$$

(iii) In the situation of (ii), the implicit equivalence of categories

$$\mathrm{Mod}_{\mathcal{D}^{\infty}_{X/Y}} \mathrm{QCoh}(X) \simeq \mathrm{Comod}_{\mathcal{J}^{\infty}_{X/Y}} \mathrm{QCoh}(X) \tag{3.163}$$

can be described as follows: The functor from left to right is given as

$$\operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} \mathcal{J}^{\infty}_{X/Y}(\mathcal{D}^{\infty}_{X/Y})^n, \tag{3.164}$$

and the functor from right to left is given as

$$\lim_{[n]\in\Delta} \mathcal{D}^{\infty}_{X/Y}(\mathcal{J}^{\infty}_{X/Y})^n. \tag{3.165}$$

Proof. (i): The functor QCoh^* satifies descent along $X \to (X/Y)_{\mathrm{str}}$. Therefore, the claim follows from Lemma 2.3.6(i).

(ii): This follows from Lemma 2.3.6(ii).

(iii): By using the explicit equivalence of categories implicit in the Barr-Beck-Lurie theorem, c.f. Lemma 2.2.4, we see that the functor from left to right is given by

$$p^*_{X/Y} \operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} p_{X/Y,!}(p^!_{X/Y} p_{X/Y,!})^n \simeq \operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} \mathcal{J}^{\infty}_{X/Y}(\mathcal{D}^{\infty}_{X/Y})^n, \tag{3.166}$$

where we used that $p^*_{X/Y}$ commutes with colimits, and that $p_{X/Y,!} \simeq p_{X/Y,*}$, by Lemma 3.2.25. Similarly, we see that the functor from right to left is given by

$$p^!_{X/Y} \lim_{[n]\in\Delta} p_{X/Y,*}(p^*_{X/Y} p_{X/Y,*})^n \simeq \lim_{[n]\in\Delta} \mathcal{D}^{\infty}_{X/Y}(\mathcal{J}^{\infty}_{X/Y})^n, \tag{3.167}$$

where we used that $p^!_{X/Y}$ commutes with limits and that $p_{X/Y,!} \simeq p_{X/Y,*}$ again. ☐

Under certain circumstances[20], we can use Lemma 3.2.27 to give formulas for the six-operations, in terms of modules over the monad $\mathcal{D}^{\infty}_X$ or comodules over the comonad $\mathcal{J}^{\infty}_X$.

[20] That is, the functors appear to exist in greater generality than the nice formulas for them do.

Theorem 3.2.28 (Formulas for the six operations)**.** *(I) Let $f : X \to Y$ be any morphism in* dAfnd*. Let $M \in \mathrm{Comod}_{\mathcal{J}_X^\infty} \mathrm{QCoh}(Y)$. The upper-star pullback*

$$f^* M \tag{3.168}$$

is naturally an object of $\mathrm{Comod}_{\mathcal{J}_X^\infty} \mathrm{QCoh}(X)$. Under the equivalence of categories (3.161)*, this operation is identified with $f_{\mathrm{str}}^* : \mathrm{Strat}(Y) \to \mathrm{Strat}(X)$.*

(II) Let $f : X \to Y$ be any morphism in dAfnd *such that $Y \to Y_{\mathrm{str}}$ is of !-descent, and let $M \in \mathrm{Comod}_{\mathcal{J}_X^\infty} \mathrm{QCoh}(X)$. Then, the object*

$$\lim_{[n]\in\Delta} \mathcal{D}_Y^\infty f_* (\mathcal{J}_X^\infty)^n M \tag{3.169}$$

is naturally an object of $\mathrm{Mod}_{\mathcal{D}_X^\infty} \mathrm{QCoh}(X)$. Under the equivalences of categories (3.161) *and* (3.162)*, this operation is identified with $f_{\mathrm{str},*} : \mathrm{Strat}(X) \to \mathrm{Strat}(Y)$.*

(III) Let $f : X \to Y$ be any morphism between objects of dAfnd *which belongs to the class of good morphisms. Assume that $X \to X_{\mathrm{str}}$ is of !-descent, and let $M \in \mathrm{Mod}_{\mathcal{D}_X^\infty} \mathrm{QCoh}(X)$. Then, the object*

$$\operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} \mathcal{J}_Y^\infty f_! (\mathcal{D}_X^\infty)^n M \tag{3.170}$$

is naturally an object of $\mathrm{Comod}_{\mathcal{J}_Y^\infty} \mathrm{QCoh}(Y)$. Under the equivalences of categories (3.162) *and* (3.161)*, this operation is identified with $f_{\mathrm{str},!} : \mathrm{Strat}(X) \to \mathrm{Strat}(Y)$.*

(IV) Let $f : X \to Y$ be any morphism between objects of dAfnd *which belongs to the class of good morphisms. Assume that $X \to X_{\mathrm{str}}$ and $Y \to Y_{\mathrm{str}}$ are of !-descent and let $M \in \mathrm{Mod}_{\mathcal{D}_Y^\infty} \mathrm{QCoh}(Y)$. The upper-shriek pullback*

$$f^! M \tag{3.171}$$

is naturally an object of $\mathrm{Mod}_{\mathcal{D}_X^\infty} \mathrm{QCoh}(X)$. Under the equivalence of categories (3.162)*, this operation is identified with $f_{\mathrm{str}}^! : \mathrm{Strat}(Y) \to \mathrm{Strat}(X)$.*

(V) Let $X \in$ dAfnd *and let $M, N \in \mathrm{Comod}_{\mathcal{J}_X^\infty} \mathrm{QCoh}(X)$. The tensor product*

$$M \widehat{\otimes}_X N \tag{3.172}$$

is naturally an object of $\mathrm{Comod}_{\mathcal{J}_X^\infty} \mathrm{QCoh}(X)$. Under the equivalence of categories (3.161)*, this operation is identified with the tensor product $\widehat{\otimes}_{X_{\mathrm{str}}}$ on $\mathrm{Strat}(X)$.*

(VI) Let $X \in$ dAfnd *and assume that $X \to X_{\mathrm{str}}$ is of !-descent. Let $M, N \in \mathrm{Mod}_{\mathcal{D}_X^\infty} \mathrm{QCoh}(X)$. Then, the object*

$$\lim_{[n]\in\Delta} \mathcal{D}_X^\infty \underline{\mathrm{Hom}}_X ((\mathcal{D}_X^\infty)^n M, N) \tag{3.173}$$

is naturally an object of $\mathrm{Mod}_{\mathcal{D}_X^\infty} \mathrm{QCoh}(X)$. Under the equivalence of categories (3.162)*, this operation is identified with the internal Hom bifunctor on $\mathrm{Strat}(X)$.*

Proof. All of these statements will be proven using Lemma 2.2.4 and the commutative (but not Cartesian) square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_Y} \\ X_{\mathrm{str}} & \xrightarrow[f_{\mathrm{str}}]{} & Y_{\mathrm{str}} \end{array} \tag{3.174}$$

in PStk. We will also freely use the equivalences (3.161) and (3.162), as well as the equivalences $p_{X,!} \simeq p_{X,*}$ and $p_{Y,!} \simeq p_{Y,*}$ coming from Lemma 3.2.25.

(I): By using Lemma 2.2.4 and Lemma 3.2.27 we know that the required functor from $\mathrm{Comod}_{\mathcal{J}_Y^\infty} \mathrm{QCoh}(Y)$ to $\mathrm{Comod}_{\mathcal{J}_X^\infty} \mathrm{QCoh}(X)$ is given by the formula

$$p_X^* f_{\mathrm{str}}^* \lim_{[n]\in\Delta} p_{Y,*}(p_Y^* p_{Y,*})^n \tag{3.175}$$

By using commutativity of the square (3.174) and Lemma 2.2.4, we have

$$p_X^* f_{\mathrm{str}}^* \lim_{[n]\in\Delta} p_{Y,*}(p_Y^* p_{Y,*})^n \simeq f^* p_Y^* \lim_{[n]\in\Delta} p_{Y,*}(p_Y^* p_{Y,*})^n \simeq f^*. \tag{3.176}$$

(II): By using Lemma 2.2.4 and Lemma 3.2.27, we know that the required functor from $\mathrm{Comod}_{\mathcal{J}_X^\infty} \mathrm{QCoh}(X)$ to $\mathrm{Mod}_{\mathcal{D}_Y^\infty} \mathrm{QCoh}(Y)$ is given by the formula

$$p_Y^! f_{\mathrm{str},*} \lim_{[n]\in\Delta} p_{X,*}(p_X^* p_{X,*})^n. \tag{3.177}$$

Using that $p_Y^!$ and $f_{\mathrm{str},*}$ commute with limits, the commutativity of the diagram (3.174), and Lemma 3.2.25 we have

$$\begin{aligned} p_Y^! f_{\mathrm{str},*} \lim_{[n]\in\Delta} p_{X,*}(p_X^* p_{X,*})^n &\simeq \lim_{[n]\in\Delta} p_Y^! f_{\mathrm{str},*} p_{X,*}(p_X^* p_{X,*})^n \\ &\simeq \lim_{[n]\in\Delta} p_Y^! p_{Y,*} f_*(p_X^* p_{X,*})^n \\ &\simeq \lim_{[n]\in\Delta} \mathcal{D}_Y^\infty f_*(\mathcal{J}_X^\infty)^n. \end{aligned} \tag{3.178}$$

(III): By using By using Lemma 2.2.4 and Lemma 3.2.27, we know that the required functor from $\mathrm{Mod}_{\mathcal{D}_X^\infty} \mathrm{QCoh}(X)$ to $\mathrm{Comod}_{\mathcal{J}_Y^\infty} \mathrm{QCoh}(Y)$ is given by the formula

$$p_Y^* f_{\mathrm{str},!} \operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} p_{X,!}(p_X^! p_{X,!})^n \tag{3.179}$$

Using that p_Y^* and $f_{\mathrm{str},!}$ commute with colimits, the commutativity of the diagram (3.174), and Lemma 3.2.25 we have

$$\begin{aligned} p_Y^* f_{\mathrm{str},!} \operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} p_{X,!}(p_X^! p_{X,!})^n &\simeq \operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} p_Y^* f_{\mathrm{str},!} p_{X,!}(p_X^! p_{X,!})^n \\ &\simeq \operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} p_Y^* p_{Y,!} f_!(p_X^! p_{X,!})^n \\ &\simeq \operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} \mathcal{J}_Y^\infty f_!(\mathcal{D}_X^\infty)^n. \end{aligned} \tag{3.180}$$

(IV): By using Lemma 2.2.4 and Lemma 3.2.27 we know that the required functor from $\mathrm{Mod}_{\mathcal{D}_Y^\infty} \mathrm{QCoh}(Y)$ to $\mathrm{Mod}_{\mathcal{D}_X^\infty} \mathrm{QCoh}(X)$ is given by the formula

$$p_X^! f_{\mathrm{str}}^! \operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} p_{Y,!}(p_Y^! p_{Y,!})^n. \tag{3.181}$$

By using the commutativity of the square (3.174) and Lemma 2.2.4, we have

$$p_X^! f_{\mathrm{str}}^! \operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} p_{Y,!}(p_Y^! p_{Y,!})^n \simeq f^! p_Y^! \operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} p_{Y,!}(p_Y^! p_{Y,!})^n \simeq f^!. \tag{3.182}$$

(V): Let $M,N \in \mathrm{Comod}_{\mathcal{J}_X^\infty}\mathrm{QCoh}(X)$. By transport of structure using Lemma 2.2.4 and Lemma 3.2.27, the tensor product on $\mathrm{Strat}(X)$ translates to the bifunctor sending M,N to

$$p_X^*\Big(\big(\lim_{[n]\in\Delta} p_{X,*}(p_X^! p_{X,*})^n M\big)\widehat{\otimes}_{X_{\mathrm{str}}}\big(\lim_{[m]\in\Delta} p_{X,*}(p_X^! p_{X,*})^m N\big)\Big). \tag{3.183}$$

Using that p_X^* is symmetric-monoidal, and Lemma 2.2.4, we have equivalences

$$\begin{aligned} & p_X^*\Big(\big(\lim_{[n]\in\Delta} p_{X,*}(p_X^! p_{X,*})^n M\big)\widehat{\otimes}_{X_{\mathrm{str}}}\big(\lim_{[m]\in\Delta} p_{X,*}(p_X^! p_{X,*})^m N\big)\Big) \\ &\simeq \big(p_X^*\lim_{[n]\in\Delta} p_{X,*}(p_X^! p_{X,*})^n M\big)\widehat{\otimes}_X\big(p_X^*\lim_{[m]\in\Delta} p_{X,*}(p_X^! p_{X,*})^m N\big) \\ &\simeq M\widehat{\otimes}_X N. \end{aligned} \tag{3.184}$$

(VI): Let $M,N \in \mathrm{Mod}_{\mathcal{D}_X^\infty}\mathrm{QCoh}(X)$. By transport of structure using Lemma 2.2.4 and Lemma 3.2.27, the internal Hom bifunctor on $\mathrm{Strat}(X)$ translates to the bifunctor sending M,N to

$$p_X^!\underline{\mathrm{Hom}}_{X_{\mathrm{str}}}\Big(\operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} p_{X,*}(p_X^! p_{X,*})^n M, \operatorname*{colim}_{[m]\in\Delta^{\mathrm{op}}} p_{X,*}(p_X^! p_{X,*})^m N\Big). \tag{3.185}$$

We have the following chain of equivalences:

$$\begin{aligned} & p_X^!\underline{\mathrm{Hom}}_{X_{\mathrm{str}}}\Big(\operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} p_{X,*}(p_X^! p_{X,*})^n M, \operatorname*{colim}_{[m]\in\Delta^{\mathrm{op}}} p_{X,*}(p_X^! p_{X,*})^m N\Big) \\ &\simeq p_X^!\lim_{[n]\in\Delta}\underline{\mathrm{Hom}}_{X_{\mathrm{str}}}\Big(p_{X,*}(p_X^! p_{X,*})^n M, \operatorname*{colim}_{[m]\in\Delta^{\mathrm{op}}} p_{X,*}(p_X^! p_{X,*})^m N\Big) \\ &\simeq p_X^!\lim_{[n]\in\Delta} p_{X,*}\underline{\mathrm{Hom}}_{X_{\mathrm{str}}}(p_{X,*}(p_X^! p_{X,*})^n M, p_X^!\operatorname*{colim}_{[m]\in\Delta^{\mathrm{op}}} p_{X,*}(p_X^! p_{X,*})^m N) \\ &\simeq p_X^!\lim_{[n]\in\Delta} p_{X,*}\underline{\mathrm{Hom}}_{X_{\mathrm{str}}}(p_{X,*}(p_X^! p_{X,*})^n M, N) \\ &\simeq \lim_{[n]\in\Delta} p_X^! p_{X,*}\underline{\mathrm{Hom}}_{X_{\mathrm{str}}}(p_{X,*}(p_X^! p_{X,*})^n M, N) \\ &\simeq \lim_{[n]\in\Delta}\mathcal{D}_X^\infty\underline{\mathrm{Hom}}_X((\mathcal{D}_X^\infty)^n M, N), \end{aligned} \tag{3.186}$$

where, in the third line we used the formula (2.90), in the fourth line we used Lemma 2.2.4, and in the fifth line we used that $p_X^!$ commutes with limits. ☐

3.2.5 The germ of the zero-section in TX

Proposition 3.2.29. *Let $X=\mathrm{Sp}(A)$ be a classical affinoid rigid space equipped with an étale morphism $X\to \mathbf{D}_K^r$. Then the algebra morphism*

$$\operatorname*{colim}_m A\widehat{\otimes}K\langle dx/p^m\rangle \to \operatorname*{colim}_{U\supset\Delta X}(A\widehat{\otimes}_K A)_U, \tag{3.187}$$

determined by $a\otimes 1\mapsto a\otimes 1$ and $dx_i \mapsto 1\otimes x_i - x_i\otimes 1$, is an isomorphism of complete bornological K-algebras. Here colimit on the left runs through the system of affinoid open neighbourhoods $U\supset\Delta X$ and $(-)_U$ denotes the corresponding affinoid localization.

Proof. Let[21] us denote the canonical morphism by

$$\varphi:\operatorname*{colim}_m A\widehat{\otimes}K\langle dx/p^m\rangle \to \operatorname*{colim}_{U\supset\Delta X}(A\widehat{\otimes}_K A)_U, \tag{3.188}$$

[21] I would like to thank Finn Wiersig for a helpful discussion about the proof of this Proposition.

in other words, for each $f = \sum_{\alpha \in \mathbf{N}^r} f_\alpha \otimes (dx)^\alpha$, one has

$$\varphi(f) = \sum_{\alpha \in \mathbf{N}^r} (f_\alpha \otimes 1)(1 \otimes x_i - x_i \otimes 1)^\alpha. \tag{3.189}$$

It is not so hard to show that φ is bounded. In order to prove the Proposition we will construct a bounded inverse ψ to φ. Let us temporarily write $x_i := x_i \otimes 1$ and $y_i := 1 \otimes x_i$. Let ∂_{x_i}, (resp. ∂_{y_i}), be the derivations on $A \widehat{\otimes}_K A$ with $\partial_{x_i}(x_j) = \delta_{ij} = \partial_{y_i}(y_j)$ and $\partial_{x_i}(y_j) = 0 = \partial_{y_i}(x_j)$ for all i, j. The multiplication map $\mu : A \widehat{\otimes}_K A \to A$ induces a morphism $\mu : \operatorname{colim}_{U \supset \Delta X} (A \widehat{\otimes}_K A)_U \to A$. Given $g \in \operatorname{colim}_{U \supset \Delta X} (A \widehat{\otimes}_K A)_U$ we set

$$\psi(g) := \sum_{\alpha \in \mathbf{N}^r} \frac{1}{\alpha!} \mu(\partial_y^\alpha g) \otimes (dx)^\alpha. \tag{3.190}$$

Now we will proceed in steps.

Step 1: The morphism ψ is well-defined and bounded. With notations as above, fix an open affinoid $U \supseteq \Delta X$. The derivations ∂_{y_i} restrict to well-defined bounded operators from $(A \otimes_K A)_U$ to itself. In particular, there exists constants C_i such that $\|\partial_{y_i} g\|_U \leqslant C_i \|g\|_U$ for all $g \in (A \otimes_K A)_U$; here $\| \cdot \|_U$ denotes the residue norm on $(A \widehat{\otimes}_K A)_U$. By the well-known fact that $|1/\alpha!| \leqslant c^\alpha$ for some $c > 0$, and by boundedness of μ, we deduce that there exists constants M and $K > 0$ such that $\left\| \frac{1}{\alpha!} \mu(\partial_y^\alpha) g \right\| \leqslant M K^\alpha \|g\|_U$; here $\| \cdot \|$ denotes the residue norm on A. Hence if $|p^N| < K^{-1}$ then ψ restricts to a bounded map $(A \otimes_K A)_U \to A \widehat{\otimes}_K K\langle dx_i / p^N \rangle_i$. This shows that ψ is well-defined and the restriction of ψ to $(A \widehat{\otimes}_K A)_U$ is bounded. Since U was arbitrary, ψ is bounded.

Step 2: The composite $\psi \circ \varphi = \mathrm{id}$. We note that the derivations ∂_{y_i} are bounded and A-linear, for the first copy of A. Hence, if $g = \varphi(f) = \sum_{\alpha \in \mathbf{N}^r} (f_\alpha \otimes 1)(y - x)^\alpha$, then $\frac{1}{\alpha!} \mu(\partial_y^\alpha g) = f_\alpha$ so that $\psi \circ \varphi = \mathrm{id}$.

Step 3: ψ is injective. We first note that we can replace the system of all affinoid neighbourhoods $\{U \supseteq \Delta X\}$ of ΔX in $X \times X$, with the system of all affinoid neighbourhoods U' of ΔX with the following property: each connected component of ΔX is contained in a unique connected component of U'. We now consider the system of algebras defining the colimit

$$\operatorname*{colim}_{U' \supseteq \Delta X} (A \widehat{\otimes}_K A)_{U'}, \tag{3.191}$$

where U' runs over the affinoid neighbourhoods as above. Recalling that X is smooth, one has in particular that each $(A \widehat{\otimes}_K A)_{U'}$ is normal. Therefore, it is a product (indexed by the connected components of U') of integral domains. It is also Noetherian. Also, by normality, all the transition morphisms in the system (3.191) are injective. We may use these facts implictly in the following.

Let $I \subseteq A \widehat{\otimes}_K A$ be the ideal defining the diagonal. Then Krull's intersection theorem for Noetherian integral domains (applied separately in each connected component of U') implies that the canonical morphism

$$(A \widehat{\otimes}_K A)_{U'} \to \lim_k (A \widehat{\otimes}_K A)_{U'} / I^k \cong \lim_k (A \widehat{\otimes}_K A) / I^k \tag{3.192}$$

is injective. Since the transition morphisms in the system (3.191) are injective we obtain a canonical injective morphism

$$\operatorname*{colim}_{U' \supseteq \Delta X} (A \widehat{\otimes}_K A)_{U'} \hookrightarrow \lim_k (A \widehat{\otimes}_K A) / I^k. \tag{3.193}$$

By flatness of A with respect to $\widehat{\otimes}_K$, we see that the morphisms in the system

$$\{A\widehat{\otimes}_K K\langle dx/p^m\rangle\}_{m\geqslant 0}. \tag{3.194}$$

are injective. Applying Krull's intersection theorem in a similar manner to above, we obtain a canonical injective morphism

$$\operatorname*{colim}_k A\widehat{\otimes}_K K\langle dx/p^m\rangle \hookrightarrow \lim_k (A\otimes K[dx])/(dx)^k. \tag{3.195}$$

These morphisms fit into a commutative square:

$$\begin{array}{ccc} \operatorname*{colim}_{U'\supseteq\Delta X}(A\widehat{\otimes}_K A) & \xrightarrow{\psi} & \operatorname*{colim}_m (A\widehat{\otimes}_K K\langle dx/p^m\rangle) \\ \big\downarrow & & \big\downarrow \\ \lim_k (A\widehat{\otimes}_K A)/I^k & \longrightarrow & \lim_k A\widehat{\otimes}_K K[dx]/(dx)^k \end{array} \tag{3.196}$$

We claim that

$$\lim_k \varphi_k : \lim_k A\widehat{\otimes}_K K[dx]/(dx)^k \to \lim_k (A\widehat{\otimes}_K A)/I^k \tag{3.197}$$

is an isomorphism, where $\varphi_k : A\widehat{\otimes}_K K[dx]/(dx)^k \to (A\widehat{\otimes}_K A)/I^k$ is induced by φ. That will be enough to prove Step 3 because the bottom arrow in (3.196) is inverse to this morphism.

Because X is smooth, the immersion $\Delta : X \to X\times X$ is regular. We recall also that $\Omega^1_{X/K}\cong I/I^2$ by sending $dx_i\mapsto y_i - x_i$. Therefore, we obtain for each $k\geqslant 0$ an isomorphism $\mathrm{Sym}^k(\Omega^1_{X/K})\to I^k/I^{k+1}$ sending $(dx)^\alpha\mapsto (y-x)^\alpha$. By passing to the graded, this implies that each φ_k is an isomorphism.

Step 4: Completing the proof. By Step 2, ψ is strict epimorphism of complete bornological spaces. It is also injective by Step 3. Therefore, ψ is an isomorphism. □

Motivated by this result we define another groupoid object as follows. In the "algebraic" setting, I found that reading [CVdB10, §4] was very useful. This discussion is also quite similar to [Cam24, Example 6.1.8]. Let X be a classical affinoid rigid space equipped with an étale morphism $X\to \mathbf{D}^r_K$. Then we can consider the *germ of the zero section* as an object of dAff:

$$(X\subseteq TX)^\dagger = \mathrm{dSp}(A\widehat{\otimes}_K K\langle dx/p^\infty\rangle). \tag{3.198}$$

There is a augmentation $\varepsilon : A\widehat{\otimes}_K K\langle dx/p^\infty\rangle \to A$ and two algebra morphisms $\sigma,\tau : A\to A\widehat{\otimes}_K K\langle dx/p^\infty\rangle$. One has

$$\sigma := \mathrm{id}\otimes 1 : A\to A\widehat{\otimes}_K K\langle dx/p^\infty\rangle, \tag{3.199}$$

which gives the *left* A-module structure on $A\widehat{\otimes}_K K\langle dx/p^\infty\rangle$. The algebra morphism τ sends a function to its Taylor series:

$$\tau(a) := \sum_{\alpha\in\mathbf{N}^r}\frac{1}{\alpha!}\partial^\alpha(a)\otimes(dx)^\alpha. \tag{3.200}$$

This gives the *right* A-module structure on $A\widehat{\otimes}_K K\langle dx/p^\infty\rangle$. There is an algebra morphism

$$\psi : A\widehat{\otimes}K\langle dx/p^\infty\rangle \to (A\widehat{\otimes}_K K\langle dx/p^\infty\rangle)\widehat{\otimes}_A(A\widehat{\otimes}_K K\langle dx/p^\infty\rangle) \tag{3.201}$$

determined by $\psi(a\otimes 1)=(a\otimes 1)\otimes(1\otimes 1)$ and

$$\psi(1\otimes dx_i)=(1\otimes dx_i)\otimes(1\otimes 1)+(1\otimes 1)\otimes(1\otimes dx_i). \tag{3.202}$$

It is important to remember that the tensor product (3.201) is taken with respect to the right A-module structure (via τ) on the first factor, and the left A-module structure on the second factor. Using the morphisms $\varepsilon,\sigma,\tau,\psi$ we obtain a groupoid object

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} (X\subseteq TX)^\dagger\times_X(X\subseteq TX)^\dagger \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} (X\subseteq TX)^\dagger \substack{\longrightarrow\\\longrightarrow} X \tag{3.203}$$

where we suppressed the degeneracy maps and we emphasise that the fiber product is taken with respect to σ and τ. Let us call this groupoid object $\exp(\mathcal{T}_X)$. Now Proposition 3.2.29 can be rephrased in the following way.

Theorem 3.2.30. *With notations as above. There is an equivalence*

$$\exp(\mathcal{T}_X)\simeq \operatorname{Inf}(X) \tag{3.204}$$

of simplicial objects in dAff.

Remark 3.2.31. *Using the isomorphism of Proposition 3.2.29 the morphisms ϵ,σ,τ and ψ may be described in the following (and more symmetric) way: ϵ is induced by the multiplication $A\widehat{\otimes}_K A\to A$, σ,τ are induced by $\mathrm{id}\otimes 1$ and $1\otimes\mathrm{id}: A\to A\widehat{\otimes}_K A$, respectively, and ψ is induced by the morphism $A\widehat{\otimes}_K A\to(A\widehat{\otimes}_K A)\widehat{\otimes}_A(A\widehat{\otimes}_K A)$ which sends $a\otimes a'\mapsto(a\otimes 1)\otimes(1\otimes a')$. We see that this is really the same as [GD67, §16.8], but we just replaced the* formal neighbourhood of the diagonal *with the* germ of the diagonal*.*

Continuing in the above setup, let derivations ∂_i be dual to dx_i. We recall that Ardakov–Wadsley's ring $\widehat{\mathcal{D}}_X(X)$ can be explicitly written as "rapidly decreasing series in the variable ∂":

$$\widehat{\mathcal{D}}_X(X)=\left\{\sum_{\alpha\in\mathbf{N}^r} f_\alpha\partial^\alpha : \|f_\alpha\| r^\alpha \xrightarrow{\alpha\to\infty} 0 \text{ for all } r>0\right\}, \tag{3.205}$$

with the expected multiplication for differential operators. For the sake of brevity let us write $J:=A\widehat{\otimes}_K K\langle dx/p^\infty\rangle$ and $U:=\widehat{\mathcal{D}}_X(X)$. We may identify J with the left A-linear (bornological) dual of U via the pairing $(\partial^\beta,(dx)^\alpha):=\alpha!\delta_{\alpha\beta}$. Using this identification one may define two[22] commuting actions of $T:=\mathcal{T}_X(X)$ on J by derivations:

$$^1\nabla_\theta(j)(D):=\theta(j(D))-j(\theta D),\quad {}^2\nabla_\theta(j)(D):=j(D\theta). \tag{3.206}$$

for $j\in J, D\in U,\theta\in T$. Using the action $^1\nabla$ of T by derivations we obtain the *de Rham complex* of J which is augmented via $\tau: A\to J$:

$$A\to\Omega^\bullet_{A/K}\widehat{\otimes}_{A,\sigma}J. \tag{3.207}$$

[22]The action $^2\nabla$ is not used in this thesis. I only mention it to emphasise that J has lots of extra structure. The actions $^1\nabla,{}^2\nabla$ can be explained in the following way. We regard J as the left A-linear dual of $U:=\widehat{\mathcal{D}}_X(X)$:

$$J=\underline{\operatorname{Hom}}_A(U,A).$$

Now U is of course a U-U-bimodule. We can view $^2\nabla$ as the naïve action by derivations on J which comes from the right U-module structure on U. On the other hand, $^1\nabla$ is the action by derivations on J which comes from the left U-module structures on U and A and Oda's rule [HTT08, Proposition 1.2.9(iii)].

Proposition 3.2.32. *The augmented de Rham complex* (3.207) *is strictly exact.*

Proof. The exactness follows from the "algebraic" version of this statement (see for instance [CVdB10, Proposition 4.2.4]), using that $A \otimes_K K[dx] \to J$ is flat. The strictness can then be obtained by applying an appropriate version of the closed-graph theorem [Wae67, §2]. □

Remark 3.2.33. *One may recognise the morphism $J \to \Omega^1_{A/K} \widehat{\otimes}_{A,\sigma} J$ in the complex* (3.207) *as being given by Euler–Lagrange operators.*

Now using this finite resolution we obtain the following.

Theorem 3.2.34. *Let $X = \mathrm{Sp}(A)$ be a classical smooth affinoid rigid space equipped with an étale morphism $X \to \mathbf{D}^r_K$. Let $p : X \to X_{\mathrm{str}}$ be the canonical morphism. Then:*

(i) $p_* 1_X \in \mathrm{QCoh}(X_{\mathrm{str}}) = \mathrm{Strat}(X)$ is descendable in the sense of Mathew [Mat16, §3.3].

(ii) The morphism $p : X \to X_{\mathrm{str}}$ is of universal !-descent. In particular, there is an equivalence of categories $\mathrm{Strat}(X) \simeq \mathrm{Mod}_{\mathcal{D}^\infty_X}(\mathrm{QCoh}(X))$, where the latter is the category of modules over the monad $\mathcal{D}^\infty_X$.

Proof. Due to Theorem 3.2.30 we may consider $X/\exp(\mathcal{T}_X) := \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} \exp(\mathcal{T}_X)_n$ with its canonical morphism $q : X \to X/\exp(\mathcal{T}_X)$ instead of $p : X \to X_{\mathrm{str}}$.

(i): Since the morphism $q : X \to X/\exp(\mathcal{T}_X)$ is of universal $*$-descent (as it is an effective epimorphism), there is an equivalence $\mathrm{QCoh}(X/\exp(\mathcal{T}_X)) \simeq \mathrm{Comod}_{q^*q_*}(\mathrm{QCoh}(X))$ induced by q^*. Therefore, to show that $1_{X/\exp(\mathcal{T})}$ belongs to the thick subcategory generated by q_*1_X, is the same as showing that 1_X belongs to the thick subcategory of $\mathrm{Comod}_{q^*q_*}(\mathrm{QCoh}(X))$ generated by $q^*q_*1_X = J$. By the Dold-Kan correspondence it is sufficient to give a bounded finite-free resolution of 1_X as a J-comodule[23]. But that is exactly Proposition 3.2.32.

(ii): By Lemma 3.2.25, we know that $q_! \simeq q_*$. By (i) above, $q_*1_X \in \mathrm{QCoh}(X/\exp(\mathcal{T}_X))$ is descendable. Therefore one may argue in an essentially identical way to [Sch22, Proposition 6.19] to deduce that q is of universal !-descent. □

3.2.6 Relation to work of Ardakov–Wadsley and Bode

Let $X = \mathrm{dSp}(A) \in \mathsf{dAfnd}$. We recall (c.f. Remark 3.2.26) that the underlying endofunctor of $\mathcal{D}^\infty_X$ can be described in this situation as

$$\mathcal{D}^\infty_X \simeq \tilde{\pi}_{2,*}\tilde{\pi}_1^! \simeq R\underline{\mathrm{Hom}}_A((A\widehat{\otimes}_K A)^\dagger_\Delta, -), \tag{3.208}$$

where $R\underline{\mathrm{Hom}}_A$ is taken with respect to the A-module structure on the first factor. It is *critically important* to note that $(A\widehat{\otimes}_K A)^\dagger_\Delta$ is an A-A bimodule. We adopt the following convention:

- ⋆ The left A-module structure on $\mathcal{D}^\infty_X M^\bullet$ (for $M^\bullet \in \mathrm{QCoh}(X)$) comes from the A-module structure on the *first factor* of $(A\widehat{\otimes}_K A)^\dagger_\Delta$,
- ⋆ The right A-module structure on $\mathcal{D}^\infty_X M^\bullet$ comes from the A-module structure on the *second factor* of $(A\widehat{\otimes}_K A)^\dagger_\Delta$.

[23] Here the A-A bimodule object J is viewed as a coalgebra under *convolution*.

As an endofunctor of $\operatorname{QCoh}(X)$, $\mathcal{D}_X^\infty(-)$ is viewed as an A-module via the *right* A-module structure. However, there may be certain situations where we wish to use the *left* A-module structure, which we will try to make clear.

The first step towards describing modules over the monad $\mathcal{D}_X^\infty$ as modules over a ring is the following.

Lemma 3.2.35. *Let A be a (derived) affinoid algebra and let $X = \operatorname{dSp}(A)$. Then the object $\mathcal{D}_X^\infty 1_X$ acquires the canonical structure of an algebra object in the ∞-category*

$$ {}_A\operatorname{BMod}_A D(\mathsf{CBorn}_K) = \operatorname{QCoh}(X \times X) \tag{3.209}$$

of A-A bimodule objects under convolution. There is a canonical morphism of monads

$$(-)\widehat{\otimes}_X \mathcal{D}_X^\infty 1_X \to \mathcal{D}_X^\infty(-). \tag{3.210}$$

We emphasise that the tensor product is taken with respect to the right *A-module structure on $\mathcal{D}_X^\infty 1_X$. The A-module structure on the left side of* (3.210) *comes from the* left *A-module structure on $\mathcal{D}_X^\infty 1_X$.*

Proof. We note that the functor $p^!$ is right adjoint to $p_!$ which is $\operatorname{QCoh}(*)$-linear. Therefore by Theorem 2.2.17, the functor $p^!$ and hence also $\mathcal{D}_X^\infty = p^! p_!$ acquires a canonical lax $\operatorname{QCoh}(*)$-linear structure. Applying the functor κ of Corollary 2.2.22, then $\mathcal{D}_X^\infty 1_X$ acquires the structure of an algebra object in the category of A-A bimodule objects under convolution. Further, the morphism (3.210) of monads is obtained from the counit of the adjunction induced by Corollary 2.2.22 on algebra objects. □

Remark 3.2.36. *One can alternatively construct the algebra structure on $\mathcal{D}_X^\infty 1_X$ via an "adjoint" Fourier–Mukai transform. Namely, the usual Fourier–Mukai transform gives a (strongly monoidal) functor*

$$\operatorname{QCoh}(X \times X) \to \operatorname{Fun}^L_{\operatorname{QCoh}(*)}(\operatorname{QCoh}(X), \operatorname{QCoh}(X)) \tag{3.211}$$

By Theorem 2.2.17 there is a strongly monoidal functor

$$\operatorname{Fun}^L_{\operatorname{QCoh}(*)}(\operatorname{QCoh}(X), \operatorname{QCoh}(X)) \to \operatorname{Fun}^{R,\mathrm{lax}}_{\operatorname{QCoh}(*)}(\operatorname{QCoh}(X), \operatorname{QCoh}(X))^{\mathsf{op}}, \tag{3.212}$$

obtained by passing to adjoints. By Corollary 2.2.22 there is an adjunction

$$\operatorname{QCoh}(X \times X) \leftrightarrows \operatorname{Fun}^{R,\mathrm{lax}}_{\operatorname{QCoh}(*)}(\operatorname{QCoh}(X), \operatorname{QCoh}(X)), \tag{3.213}$$

in which the left adjoint is strongly monoidal (for convolution), hence the right adjoint is canonically lax monoidal. The object $(A\widehat{\otimes}_K A)^\dagger_\Delta \in \operatorname{QCoh}(X \times X)$ is a coalgebra under convolution. The image of this object under the composite of (3.211), (3.212) *and the right adjoint in* (3.213) *gives the object $\mathcal{D}_X^\infty 1_X \in \operatorname{QCoh}(X \times X)$ together with its algebra object structure (with respect to convolution).*

The main point of the rest of this section is to identify a class of objects such that the natural transformation (3.210) restricts to an equivalence on such objects. Furthermore, the class of such objects should be large enough to include the examples of interest, e.g. the underlying A-modules of $\mathcal{C}$-complexes [Bod21, §8], and be preserved by the monad $\mathcal{D}_X^\infty$. For this purpose we introduce the following definition.

Definition 3.2.37. *Let $A \in \mathsf{dAfnd}$ and let $X = \operatorname{dSp}(A)$.*

(i) We define

$$\mathrm{Fr}(X) \subseteq \mathrm{QCoh}(X) \tag{3.214}$$

to be the full sub-∞-category spanned by those objects $M^\bullet$ whose underlying object $M^\bullet \in D(\mathsf{CBorn}_K)$ is such that, for each $j \in \mathbf{Z}$, $H^j(M^\bullet)$ is a Fréchet space.

(ii) We define

$$\mathrm{sFr}(X) \subseteq \mathrm{Fr}(X) \tag{3.215}$$

to be the full sub-∞-category spanned by those objects $M^\bullet$ such that $A_U \widehat{\otimes}^{\mathbf{L}}_A M^\bullet \in \mathrm{Fr}(U)$ for every affinoid subdomain $U \subseteq X$. We may refer to such objects as stably Fréchet *complexes.*

Before proceeding further we fix notations as in §3.2.5. If $X = \mathrm{Sp}(A)$ is a classical affinoid rigid space equipped with an étale morphism $X \to \mathbf{D}^r_K$, we let $x_1, \dots, x_r \in A$ be the corresponding étale coordinates. For every $0 \leqslant m < \infty$ and we define the K-linear pairing

$$K\langle dx/p^m \rangle \times K\langle p^m \partial \rangle \to K \tag{3.216}$$

by $((dx)^\alpha, \partial^\beta) := \alpha! \delta_{\alpha\beta}$, for every pair of multi-indices $\alpha, \beta \in \mathbf{N}^r$. We define $K\langle dx/p^\infty \rangle := \mathrm{colim}_m K\langle dx/p^m \rangle$ and $K\langle p^\infty \partial \rangle := \lim_m K\langle p^m \partial \rangle$ where the (co)limits are taken in CBorn_K.

Lemma 3.2.38. *If $V \in \mathsf{CBorn}_K$ is a Fréchet space, then the canonical morphism*

$$V \widehat{\otimes}^{\mathbf{L}}_K K\langle p^\infty \partial \rangle \to R\underline{\mathrm{Hom}}_K(K\langle dx/p^\infty \rangle, V) \tag{3.217}$$

is an equivalence in $D(\mathsf{CBorn}_K)$. Further, the natural morphism

$$V \widehat{\otimes}^{\mathbf{L}}_K K\langle p^\infty \partial \rangle \to V \widehat{\otimes}_K K\langle p^\infty \partial \rangle \tag{3.218}$$

is an equivalence, so that both sides are concentrated in degree 0.

Proof. By Lemma 2.1.30 V can be presented as an $\aleph_1$-filtered colimit

$$V \simeq \operatorname*{colim}_{[B] \in \mathfrak{S}(V)} V_B \tag{3.219}$$

of Banach spaces. By Corollary 2.1.24 this is even the colimit in $D(\mathsf{CBorn}_K)$. Using this together with the fact that we can exchange countable limits with $\aleph_1$-filtered colimits (by Proposition 2.1.49 and Lemma 2.1.42), we obtain

$$\begin{aligned} R\underline{\mathrm{Hom}}_K(K\langle dx/p^\infty \rangle, V) &\simeq R\lim_n L\operatorname*{colim}_{[B]} \underline{\mathrm{Hom}}_K(K\langle dx/p^n \rangle, V_B) \\ &\simeq L\operatorname*{colim}_{[B]} R\lim_n \underline{\mathrm{Hom}}_K(K\langle dx/p^n \rangle, V_B). \end{aligned} \tag{3.220}$$

Now cofinality together with the Mittag-Leffler result of [Bod21, Theorem 5.24] implies that

$$R\lim_n \underline{\mathrm{Hom}}_K(K\langle dx/p^n \rangle, V_B) \simeq R\lim_n (V_B \widehat{\otimes}_K K\langle p^n \partial \rangle) \simeq \lim_n (V_B \widehat{\otimes}_K K\langle p^n \partial \rangle). \tag{3.221}$$

viewed as an object in degree 0. Next we note that

$$V_B \widehat{\otimes}_K K\langle p^\infty \partial \rangle \cong \lim_n (V_B \widehat{\otimes}_K K\langle p^n \partial \rangle), \tag{3.222}$$

in CBorn_K, because both sides can be written as rapidly decreasing series with coefficients in the Banach space V_B. Because $K\langle p^\infty \partial \rangle$ is *strongly flat* [Bod21, Corollary 5.36], we obtain $V_B \widehat{\otimes}_K K\langle p^\infty \partial \rangle \simeq V_B \widehat{\otimes}^{\mathbf{L}}_K K\langle p^\infty \partial \rangle$. Hence we may conclude by using that $\widehat{\otimes}^{\mathbf{L}}_K$ is compatible with colimits (separately in each variable). ☐

Proposition 3.2.39. *Let $X = \mathrm{Sp}(A)$ be a classical affinoid equipped with an étale morphism $X \to \mathbf{D}^r_K$. There is an equivalence $\mathcal{D}^\infty_X 1_X \simeq \widehat{\mathcal{D}}_X(X)$ of algebra objects in* $\mathrm{QCoh}(X)$.

Proof. Let $q : X \to \mathrm{X} / \exp(\mathcal{T}_X)$ be the canonical morphism. Thanks to Theorem 3.2.30 and the construction of Lemma 3.2.35 we know that there is an equivalence of algebra objects[24] $\mathcal{D}^\infty_X 1_X \simeq q^! q_! 1_X$. Now by base-change one has

$$q^! q_! 1_X \simeq R\underline{\mathrm{Hom}}_A(A \widehat{\otimes}_K K\langle dx/p^\infty\rangle, A) \tag{3.223}$$

and using Lemma 3.2.38 above we know that

$$\widehat{\mathcal{D}}_X(X) \simeq A \widehat{\otimes}_K K\langle p^\infty \partial\rangle \simeq R\underline{\mathrm{Hom}}_A(A \widehat{\otimes}_K K\langle dx/p^\infty\rangle, A), \tag{3.224}$$

as left A-modules, so we deduce that $q^! q_! 1_X$ is (strongly) flat and concentrated in degree 0. In particular it comes from an algebra object of the ordinary category ${}_A\,\mathrm{BMod}_A\, \mathsf{CBorn}_K$. We recall that the isomorphism (3.224) identifies each ∂^α with the A-linear map determined by

$$\partial^\alpha((dx)^\beta) = \alpha! \delta_{\alpha\beta}. \tag{3.225}$$

Now we use notations as in §3.2.5, in particular, we recall the definitions of the algebra morphisms $\varepsilon, \sigma, \tau$ and ψ. We regard $\underline{\mathrm{Hom}}_A(A \widehat{\otimes}_K K\langle dx/p^\infty\rangle, A)$ as a left A-module via $(a.\eta)(-) := \eta(\sigma(a) \cdot -)$ and as a right A-module via $(\eta.a)(-) := \eta(\tau(a) \cdot -)$, for $a \in A$ and $\eta \in \underline{\mathrm{Hom}}_A(A \widehat{\otimes}_K K\langle dx/p^\infty\rangle, A)$. Under the identification (3.225), the right and left actions become

$$a.\partial^\alpha = a\partial^\alpha \quad \text{and} \quad \partial^\alpha.a = \sum_{\beta+\gamma=\alpha} \binom{\beta}{\alpha} \partial^\beta(a)\partial^\gamma. \tag{3.226}$$

We need to check that the composite

$$\begin{aligned} &\underline{\mathrm{Hom}}_A(A \widehat{\otimes}_K K\langle dx/p^\infty\rangle, A) \widehat{\otimes}_A \underline{\mathrm{Hom}}_A(A \widehat{\otimes}_K K\langle dx/p^\infty\rangle, A) \\ &\qquad \to \underline{\mathrm{Hom}}_A((A \widehat{\otimes}_K K\langle dx/p^\infty\rangle) \widehat{\otimes}_A (A \widehat{\otimes}_K K\langle dx/p^\infty\rangle), A) \\ &\qquad\qquad \xrightarrow{\psi^\vee} \underline{\mathrm{Hom}}_A(A \widehat{\otimes}_K K\langle dx/p^\infty\rangle, A) \end{aligned} \tag{3.227}$$

agrees with the multiplication on $\widehat{\mathcal{D}}_X(X)$. In the first line, the tensor product is taken with respect to the right A-module structure on the first factor and the left A-module structure on the second factor. To be completely explicit, the first morphism sends $\eta \otimes \eta'$ to the morphism $\eta \widetilde{\otimes} \eta'$ determined by

$$(\eta \widetilde{\otimes} \eta')(j \otimes j') := (\eta.\eta'(j'))(j) = \eta(\tau(\eta'(j'))j), \tag{3.228}$$

for $j, j' \in A \widehat{\otimes}_K K\langle dx/p^\infty\rangle$. One checks that

$$\partial^\alpha \widetilde{\otimes} \partial^\beta((1 \otimes dx)^\gamma \otimes (1 \otimes dx)^\epsilon) = \alpha!\beta!\delta_{\alpha\gamma}\delta_{\beta\epsilon}. \tag{3.229}$$

Let us denote the composite (3.227) by m. It follows from (3.229) that

$$\begin{aligned} m(\partial^\alpha \otimes \partial^\beta)((1 \otimes dx)^\gamma) &= (\partial^\alpha \widetilde{\otimes} \partial^\beta)(\psi((1 \otimes dx)^\gamma)) \\ &= (\partial^\alpha \widetilde{\otimes} \partial^\beta)\Big(\binom{\gamma}{\delta}(1 \otimes dx)^\delta \otimes (1 \otimes dx)^\epsilon\Big) \\ &= \gamma! \delta_{\alpha+\beta,\gamma} \\ &= \partial^{\alpha+\beta}((1 \otimes dx)^\gamma), \end{aligned} \tag{3.230}$$

[24] In the category of A-A bimodule objects with the convolution monoidal structure.

so that $m(\partial^\alpha \otimes \partial^\beta) = \partial^{\alpha+\beta}$. Using this together with the fact that m is balanced and left A-linear, one has

$$\begin{aligned} m(f\partial^\alpha \otimes g\partial^\beta) &= m((f.\partial^\alpha.g) \otimes \partial^\alpha) \\ &= m\Big(\sum_{\eta+\nu=\alpha} \binom{\alpha}{\eta} f\partial^\eta(g)\partial^\nu \otimes \partial^\beta \Big) \\ &= \sum_{\eta+\nu=\alpha} \binom{\alpha}{\eta} f\partial^\eta(g)\partial^{\nu+\beta}. \end{aligned} \tag{3.231}$$

Therefore $m(f\partial^\alpha \otimes g\partial^\beta) = \sum_{\eta+\nu=\alpha}\binom{\alpha}{\eta} f\partial^\eta(g)\partial^{\nu+\beta}$ agrees with the multiplication on $\widehat{\mathcal{D}}_X(X)$, as required. □

Theorem 3.2.40. *Suppose that $X = \mathrm{Sp}(A)$ is a smooth classical affinoid equipped with an étale morphism $X \to \mathbf{D}^r_K$. Let $M^\bullet \in \mathrm{Fr}(X)$. Then the canonical morphism*

$$M^\bullet \widehat{\otimes}_X \mathcal{D}^\infty_X 1_X \to \mathcal{D}^\infty_X M^\bullet \tag{3.232}$$

of Lemma 3.2.35, is an equivalence.

Proof. Using Theorem 3.2.30, what we need to show is that the canonical morphism

$$M^\bullet \widehat{\otimes}^{\mathbf{L}}_A R\underline{\mathrm{Hom}}_K(K\langle dx/p^\infty\rangle, A) \to R\underline{\mathrm{Hom}}_K(K\langle dx/p^\infty\rangle, M^\bullet) \tag{3.233}$$

is an equivalence in $D(\mathsf{CBorn}_K)$. The right side of (3.233) is equivalent to

$$R\lim_n \underline{\mathrm{Hom}}_K(K\langle dx/p^n\rangle, M^\bullet), \tag{3.234}$$

and therefore by [Bod21, Lemma 3.3], for each $j \in \mathbf{Z}$ we obtain a short-exact sequence

$$\begin{aligned} 0 \to R^1\lim_n \underline{\mathrm{Hom}}_K(K\langle dx/p^n\rangle, H^{j-1}(M^\bullet)) \\ \to H^j(R\underline{\mathrm{Hom}}_K(K\langle dx/p^\infty\rangle, M^\bullet)) \\ \to \lim_n \underline{\mathrm{Hom}}_K(K\langle dx/p^n\rangle, H^j(M^\bullet)) \to 0. \end{aligned} \tag{3.235}$$

Now by Lemma 3.2.38, the first term is zero and the second term is isomorphic to

$$H^j(M^\bullet)\widehat{\otimes}_K K\langle p^\infty \partial\rangle \cong H^j(M^\bullet \widehat{\otimes}^{\mathbf{L}}_K K\langle p^\infty\partial\rangle), \tag{3.236}$$

where we again used [Bod21, Corollary 5.36]. This shows that (3.233) is an isomorphism after taking cohomology, and therefore an equivalence. □

Proposition 3.2.41. *Let $X = \mathrm{Sp}(A)$ be a smooth classical affinoid which admit an étale morphism to a polydisk. Then the endofunctors $\mathcal{D}^\infty_X(-)$ and $(-)\widehat{\otimes}_X \mathcal{D}^\infty_X 1_X$ preserve the full subcategories $\mathrm{Fr}(X)$ and $\mathrm{sFr}(X)$ of $\mathrm{QCoh}(X)$.*

Proof. Let $M^\bullet \in \mathrm{QCoh}(X)$. After forgetting to $D(\mathsf{CBorn}_K)$, the object $M^\bullet \widehat{\otimes}^{\mathbf{L}}_X \mathcal{D}^\infty_X 1_X$ is nothing but

$$M^\bullet \widehat{\otimes}^{\mathbf{L}}_A A \widehat{\otimes}^{\mathbf{L}}_K K\langle p^\infty\partial\rangle \simeq M^\bullet \widehat{\otimes}^{\mathbf{L}}_K K\langle p^\infty\partial\rangle. \tag{3.237}$$

where we used [Bod21, Corollary 5.36], and by the result of *loc. cit.* again one has $H^j(M^\bullet \widehat{\otimes}^{\mathbf{L}}_K K\langle p^\infty\partial\rangle) \cong H^j(M^\bullet)\widehat{\otimes}_K K\langle p^\infty\partial\rangle$. This proves that $(-)\widehat{\otimes}_X \mathcal{D}^\infty_X 1_X$ preserves

$\mathrm{Fr}(X)$. In order to show that $(-)\widehat{\otimes}_X\mathcal{D}_X^\infty 1_X$ preserves $\mathrm{sFr}(X) \subseteq \mathrm{Fr}(X)$ it then suffices to show that the natural morphism $\mathcal{D}_X^\infty 1_X \widehat{\otimes}_A^{\mathbf{L}} A_U \to \mathcal{D}_U^\infty 1_U$ is an equivalence, for each affinoid subdomain $U \subseteq X$. This can be deduced (for instance) from Proposition 3.2.39, because $\widehat{\mathcal{D}}_X(X) \xrightarrow{\sim} \widehat{\mathcal{D}}_U(U)\widehat{\otimes}_A^{\mathbf{L}} A_U$. □

Corollary 3.2.42. *Let $X = \mathrm{Sp}(A)$ be a smooth classical affinoid equipped with an étale morphism $X \to \mathbf{D}_K^r$. The morphism* (3.210) *restricts to an equivalence of monads on* $\mathrm{Fr}(X)$. *Consequently, there is an equivalence of ∞-categories*

$$\mathrm{RMod}_{\mathcal{D}_X^\infty 1_X} \mathrm{Fr}(X) \simeq \mathrm{Mod}_{\mathcal{D}_X^\infty} \mathrm{Fr}(X). \tag{3.238}$$

The same holds with $\mathrm{sFr}(X)$ *in place of* $\mathrm{Fr}(X)$.

Proof. Follows by assembling Theorem 3.2.40 and Proposition 3.2.41. □

In the remainder of this subsection $X = \mathrm{Sp}(A)$ denotes a classical affinoid equipped with an étale morphism $X \to \mathbf{D}_K^r$. We recall the definition of the Banach completed differential operators $\mathcal{D}_X^n(X)$ from [Bod21, §2]. These are Noetherian Banach algebras and $\widehat{\mathcal{D}}_X(X) = \lim_n \mathcal{D}_X^n(X)$ gives a presentation of $\widehat{\mathcal{D}}_X(X)$ as a Fréchet–Stein algebra.

Definition 3.2.43. *An object $M^\bullet \in \mathrm{RMod}_{\widehat{\mathcal{D}}_X(X)} D(\mathsf{CBorn}_K)$ is called a* $\mathcal{C}$-complex *if:*

(i) each $M_n^\bullet := M^\bullet \widehat{\otimes}_{\widehat{\mathcal{D}}_X(X)}^{\mathbf{L}} \mathcal{D}_X^n(X)$ is such that each $H^j(M_n^\bullet)$ is a finitely-generated $\mathcal{D}_X^n(X)$-module and $H^j(M_n^\bullet) = 0$ for $|j| \gg 0$;

(ii) the canonical morphism $M^\bullet \to R\lim_n M_n^\bullet$ is an equivalence.

We denote the full subcategory spanned by such objects, by $D_{\mathcal{C}}(X)$.

Remark 3.2.44. *By [Bod21, Lemma 8.11], condition (ii) in Definition 3.2.43 can be replaced with the following (which may be easier to check in practice):*

(ii)′ for each $j \in \mathbf{Z}$ the canonical morphism $H^j(M^\bullet) \to \lim_n H^j(M_n^\bullet)$ is an isomorphism.

Lemma 3.2.45. *Suppose that $M^\bullet$ is a $\mathcal{C}$-complex. Then the underlying object $M^\bullet \in$* $\mathrm{QCoh}(X)$ *belongs to the full subcategory* $\mathrm{Fr}(X) \subseteq \mathrm{QCoh}(X)$, *so that one has an inclusion*

$$D_{\mathcal{C}}(X) \subseteq \mathrm{RMod}_{\widehat{\mathcal{D}}_X(X)} \mathrm{Fr}(X). \tag{3.239}$$

Proof. This is clear from Remark 3.2.44. □

Assembling all the above together with Theorem 3.2.34 we may draw the following diagram relating various categories.

$$\begin{array}{ccc}
 & & \mathrm{Strat}(X) \\
 & & \big\downarrow \simeq \\
\mathrm{Mod}_{\mathcal{D}_X^\infty} \mathrm{Fr}(X) & \hookrightarrow & \mathrm{Mod}_{\mathcal{D}_X^\infty} \mathrm{QCoh}(X) \\
\big\downarrow \simeq & & \\
\mathrm{RMod}_{\mathcal{D}_X^\infty 1_X} \mathrm{Fr}(X) & \hookrightarrow & \mathrm{RMod}_{\mathcal{D}_X^\infty 1_X} \mathrm{QCoh}(X) \\
\big\downarrow \simeq & & \big\downarrow \simeq \\
\mathrm{RMod}_{\widehat{\mathcal{D}}_X(X)} \mathrm{Fr}(X) & \hookrightarrow & \mathrm{RMod}_{\widehat{\mathcal{D}}_X(X)} \mathrm{QCoh}(X) \\
\big\uparrow & & \\
D_{\mathcal{C}}(X) & &
\end{array} \tag{3.240}$$

In particular we obtain the following.

Theorem 3.2.46. *Let $X = \mathrm{Sp}(A)$ be a smooth classical affinoid equipped with an étale morphism $X \to \mathbf{D}^r_K$. Then there is a fully-faithful functor of ∞-categories*

$$D_{\mathcal{C}}(X) \hookrightarrow \mathrm{Strat}(X). \tag{3.241}$$

3.2.7 Descent for $\widehat{\mathcal{D}}$-modules

In this section we continue to let $X = \mathrm{Sp}(A)$ be a smooth affinoid equipped with an étale morphism $X \to \mathbf{D}^r_K$. Let X_w (resp. $X_{n,w}$) denote the poset of affinoid subdomains (resp. p^n-accessible subdomains[25]) of X. By using [Lur17, Proposition 4.6.2.17] and unstraightening we obtain functors

$$\begin{aligned} \mathrm{RMod}_{\widehat{\mathcal{D}}_X(-)}\, D(\mathsf{CBorn}_K) &: X_w^{\mathsf{op}} \to \mathsf{Cat}_\infty \\ \mathrm{RMod}_{\mathcal{D}^n_X(-)}\, D(\mathsf{CBorn}_K) &: X_{n,w}^{\mathsf{op}} \to \mathsf{Cat}_\infty \end{aligned} \tag{3.242}$$

which send $t : V \hookrightarrow U$ to the pullback functors $(-)\widehat{\otimes}^{\mathbf{L}}_{\widehat{\mathcal{D}}_X(U)}\widehat{\mathcal{D}}_X(V)$ and $(-)\widehat{\otimes}^{\mathbf{L}}_{\mathcal{D}^n_X(U)}\mathcal{D}^n_X(V)$ respectively. We recall that $\widehat{\mathcal{D}}_X$ (resp. $\mathcal{D}^n_X(X)$) is a sheaf of algebras on X_w (resp. $X_{n,w}$). More precisely, we recall that Ardakov and Wadsley have proved the counterpart of Tate acyclicity for this Fréchet-Stein algebra.

Proposition 3.2.47. *Let X_w (resp. $X_{n,w}$) be the poset of affinoid subdomains (resp. p^n-accessible subdomains) of X equipped with the* weak G-topology. *Then:*

(i) [AW19, §8.1] Let $\{U_i \to X\}_{i=1}^s$ be a covering in X_w. Then the augmented (alternating) Čech complex $C^\bullet_{\mathrm{aug}}(\{U_i\}, \widehat{\mathcal{D}}_X)$ is exact.

(ii) [AW19, Theorem 3.5] Let $\{U_i \to X\}_{i=1}^s$ be a covering in $X_{w,n}$. Then the augmented (alternating) Čech complex $C^\bullet_{\mathrm{aug}}(\{U_i\}, \mathcal{D}^n_X)$ is exact.

Remark 3.2.48. *This immediately implies that $C^\bullet_{\mathrm{aug}}(\{U_i\}, \widehat{\mathcal{D}}_X)$ (resp. $C^\bullet_{\mathrm{aug}}(\{U_i\}, \mathcal{D}^n_X)$) is strictly exact: as it is a complex of Fréchet (resp. Banach) spaces, we can appeal to the open mapping theorem.*

Lemma 3.2.49. *(i) Let $U, V \subseteq X$ be affinoid subdomains. Then the canonical morphism*

$$\widehat{\mathcal{D}}_X(U)\widehat{\otimes}^{\mathbf{L}}_{\widehat{\mathcal{D}}_X(X)}\widehat{\mathcal{D}}_X(V) \to \widehat{\mathcal{D}}_X(U \cap V) \tag{3.243}$$

is an equivalence of $\widehat{\mathcal{D}}_X(U)$-$\widehat{\mathcal{D}}_X(V)$ bimodule objects in $D(\mathsf{CBorn}_K)$.

(ii) Let $U, V \subseteq X$ be p^n-accessible affinoid subdomains. Then the canonical morphism

$$\mathcal{D}^n_X(U)\widehat{\otimes}^{\mathbf{L}}_{\mathcal{D}^n_X(X)}\mathcal{D}^n_X(V) \to \mathcal{D}^n_X(U \cap V) \tag{3.244}$$

is an equivalence of $\mathcal{D}^n_X(U)$-$\mathcal{D}^n_X(V)$ bimodule objects in $D(\mathsf{CBorn}_K)$.

Proof. We only prove (i) as the proof of (ii) is very similar. Let $A_U, A_V, A_{U\cap V}$ denote the corresponding affinoid localizations. Using the isomorphism $\widehat{\mathcal{D}}_X(U) \simeq A_U\widehat{\otimes}^{\mathbf{L}}_K K\langle p^\infty \partial\rangle$,

[25] By this we mean $p^n\mathcal{T}_X$-accessible, in the sense of [AW19, §4.5].

and associativity properties of the (derived) tensor product, the morphism is equivalent to the morphism

$$A_U \widehat{\otimes}^{\mathbf{L}}_A A_V \widehat{\otimes}^{\mathbf{L}}_K K\langle p^\infty \partial \rangle \to A_{U\cap V} \widehat{\otimes}^{\mathbf{L}}_K K\langle p^\infty \partial \rangle, \tag{3.245}$$

which evidently comes from $A_U \widehat{\otimes}^{\mathbf{L}}_A A_V \to A_{U\cap V}$ by tensoring on the right. The latter is an equivalence by [BBK17, Theorem 5.16]. □

Lemma 3.2.50. *With notations as above. Let $\{U_i \to X\}_{i=1}^s$ be an admissible covering of X by affinoid subdomains. Then:*

(i) The morphism $\widehat{\mathcal{D}}_X(X) \to \prod_{i=1}^s \widehat{\mathcal{D}}_X(U_i)$ is descendable in the sense of Definition 2.2.8.

(ii) Let $Y := \coprod_{i=1}^s U_i$. Then the augmented simplicial object $\widehat{\mathcal{D}}_X(Y^{\bullet+1/X})$ satisfies the Beck-Chevalley condition of Definition 2.2.13.

(iii) The canonical morphism

$$\mathrm{RMod}_{\widehat{\mathcal{D}}_X(X)} D(\mathsf{CBorn}_K) \to \lim_{[m]\in\Delta} \mathrm{RMod}_{\widehat{\mathcal{D}}_X(Y^{m+1/X})} D(\mathsf{CBorn}_K) \tag{3.246}$$

is an equivalence of ∞-categories. In particular $\mathrm{RMod}_{\widehat{\mathcal{D}}_X(-)} D(\mathsf{CBorn}_K)$ is a sheaf on X_w.

Proof. (i): By combining Proposition 3.2.47 with Lemma 3.2.49, and using the Dold-Kan correspondence, one deduces that

$$\widehat{\mathcal{D}}_X(X) \to R\lim\Big(\prod_i \widehat{\mathcal{D}}_X(U_i) \rightrightarrows \prod_{i<j} \widehat{\mathcal{D}}_X(U_i) \widehat{\otimes}^{\mathbf{L}}_{\widehat{\mathcal{D}}_X(X)} \widehat{\mathcal{D}}_X(U_j) \rightrightarrows \cdots \Big),$$

as $\widehat{\mathcal{D}}_X(X)$-$\widehat{\mathcal{D}}_X(X)$ bimodule objects in $D(\mathsf{CBorn}_K)$. We note that the limit on the right is finite because we used the alternating Čech complex. This establishes (i).

(ii): This is immediate from Lemma 3.2.49.

(iii): By using (i) and (ii) above, this follows from Lemma 2.2.14. □

In an entirely similar way one has the following.

Lemma 3.2.51. *With notations as above. Let $\{U_i \to X\}_{i=1}^s$ be an admissible covering of X by p^n-accessible affinoid subdomains. Then:*

(i) The morphism $\mathcal{D}^n_X(X) \to \prod_{i=1}^s \mathcal{D}^n_X(U_i)$ is descendable in the sense of Definition 2.2.8.

(ii) Let $Y := \coprod_{i=1}^s U_i$. Then the augmented simplicial object $\mathcal{D}^n_X(Y^{\bullet+1/X})$ satisfies the Beck-Chevalley condition of Definition 2.2.13.

(iii) The canonical morphism

$$\mathrm{RMod}_{\mathcal{D}^n_X(X)} D(\mathsf{CBorn}_K) \to \lim_{[m]\in\Delta} \mathrm{RMod}_{\mathcal{D}^n_X(Y^{m+1/X})} D(\mathsf{CBorn}_K) \tag{3.247}$$

is an equivalence of ∞-categories. In particular $\mathrm{RMod}_{\mathcal{D}^n_X(-)} D(\mathsf{CBorn}_K)$ is a sheaf on $X_{n,w}$.

Proof. This is the same, *mutandis mutatis*, as the proof of Lemma 3.2.50. □

For each $n \geqslant 0$ we denote by

$$\mathrm{RMod}^{\mathrm{b,fg}}_{\mathcal{D}^n_X(X)}\, D(\mathsf{CBorn}_K) \subseteq \mathrm{RMod}_{\mathcal{D}^n_X(X)}\, D(\mathsf{CBorn}_K) \tag{3.248}$$

the full subcategory spanned by (cohomologically) bounded complexes with finitely-generated cohomology groups.

Lemma 3.2.52. *Let $U \subseteq X$ be a p^n-accessible affinoid subdomain. Then:*

(i) Finitely-generated right $\mathcal{D}^n_X(X)$-modules are acyclic for $(-)\widehat{\otimes}_{\mathcal{D}^n_X(X)}\mathcal{D}^n_X(U)$.

(ii) The pullback functor $(-)\widehat{\otimes}^{\mathbf{L}}_{\mathcal{D}^n_X(X)}\mathcal{D}^n_X(U)$ restricts to a functor

$$\mathrm{RMod}^{\mathrm{b,fg}}_{\mathcal{D}^n_X(X)}\, D(\mathsf{CBorn}_K) \to \mathrm{RMod}^{\mathrm{b,fg}}_{\mathcal{D}^n_X(U)}\, D(\mathsf{CBorn}_K), \tag{3.249}$$

which furthermore is t-exact. In particular we obtain a sub-prestack

$$\mathrm{RMod}^{\mathrm{b,fg}}_{\mathcal{D}^n_X(-)}\, D(\mathsf{CBorn}_K) \subseteq \mathrm{RMod}_{\mathcal{D}^n_X(-)}\, D(\mathsf{CBorn}_K) : X_w^{\mathsf{op}} \to \mathsf{Cat}_\infty. \tag{3.250}$$

Proof. (i): By [AW19, Theorem 4.9], $\mathcal{D}^n_X(U)$ is flat on both sides as an abstract $\mathcal{D}^n_X(X)$-module. Hence the claim follows from [Bod21, Lemma 5.32], noting that the Tor-groups in *loc. cit.* refer to the abstract Tor-groups.

(ii): Let $M^\bullet$ be a bounded complex with finitely-generated cohomology groups. Using the result of (i), an easy spectral sequence argument implies that

$$H^j(M^\bullet \widehat{\otimes}^{\mathbf{L}}_{\mathcal{D}^n_X(X)}\mathcal{D}^n_X(U)) \cong H^j(M^\bullet)\widehat{\otimes}_{\mathcal{D}^n_X(X)}\mathcal{D}^n_X(U), \tag{3.251}$$

proving the Lemma. ☐

Theorem 3.2.53. *Let $\{U_i \to X\}_{i=1}^s$ be an admissible covering of X by p^n-accessible subdomains. Then the canonical morphism*

$$\mathrm{RMod}^{\mathrm{b,fg}}_{\mathcal{D}^n_X(X)}\, D(\mathsf{CBorn}_K) \to \lim_{[m]\in\Delta} \mathrm{RMod}^{\mathrm{b,fg}}_{\mathcal{D}^n_X(Y^{m+1/X})}\, D(\mathsf{CBorn}_K) \tag{3.252}$$

is an equivalence of ∞-categories.

Proof. Let $(M^\bullet_m)_{[m]\in\Delta}$ be an object of the right-hand side of (3.252). Because of Lemma 3.2.51, the only thing to show is that $M^\bullet_{-1} := R\lim_{[m]\in\Delta} M^\bullet_m$ is bounded with finitely-generated cohomology groups. We consider the hypercohomology spectral sequence

$$E_2^{pq} : R^p \lim_{[m]\in\Delta} H^q M^\bullet_m \Rightarrow H^{p+q}(M_{-1}). \tag{3.253}$$

This converges because $(M^\bullet_m)_{[m]\in\Delta}$ is uniformly cohomologically bounded by Lemma 3.2.52. Further, Lemma 3.2.52 implies that

$$H^q(M^\bullet_k)\widehat{\otimes}_{\mathcal{D}^n_X(Y^{k+1/X})}\mathcal{D}^n_X(Y^{l+1/X}) \cong H^q(M^\bullet_l) \tag{3.254}$$

is an isomorphism for every cosimplicial morphism $[l] \to [k]$. Thus the counterpart of Kiehl's theorem in this setting [AW19, §5] then implies that

$$R^p \lim_{[m]\in\Delta} H^q M^\bullet_m = 0, \text{ whenever } p > 0. \tag{3.255}$$

Thus the spectral sequence (3.253) collapses and gives an isomorphism

$$\begin{aligned} H^n M^\bullet_{-1} &\cong \lim_{[m]\in\Delta} H^n(M^\bullet_m) \\ &= \mathrm{eq}\left(H^n(M^\bullet_0) \rightrightarrows H^n(M^\bullet_1)\right) \end{aligned} \tag{3.256}$$

which, by the theorem of descent for finitely-generated $\mathcal{D}^n_X$-modules [AW19, §5], is a finitely generated $\mathcal{D}^n_X(X)$-module. Further, we see that the cohomological amplitude of $M^\bullet_{-1}$ is contained in the same interval as $M^\bullet_0$. ☐

Scholium 3.2.54. *Looking at the proof of Theorem 3.2.53, we notice that in fact*

$$\mathrm{RMod}^{[c,d],\mathrm{fg}}_{\mathcal{D}^n_X(X)} D(\mathsf{CBorn}_K) \to \lim_{[m]\in\Delta} \mathrm{RMod}^{[c,d],\mathrm{fg}}_{\mathcal{D}^n_X(Y^{m+1/X})} D(\mathsf{CBorn}_K) \tag{3.257}$$

is an equivalence of ∞-categories, for any interval $[c,d] \subseteq \mathbf{R}$ with $d < \infty$.

We remark that since each $\mathcal{D}^n_X(X)$ is flat (on both sides) as an abstract $\mathcal{D}^{n+1}_X(X)$-module the pullback functors restrict[26] to functors

$$\mathrm{RMod}^{\mathrm{b,fg}}_{\mathcal{D}^{n+1}_X(X)} D(\mathsf{CBorn}_K) \to \mathrm{RMod}^{\mathrm{b,fg}}_{\mathcal{D}^n_X(X)} D(\mathsf{CBorn}_K) \tag{3.258}$$

for each n. There is an obvious functor

$$\phi : D_{\mathcal{C}}(X) \to \lim_n \mathrm{RMod}^{\mathrm{b,fg}}_{\mathcal{D}^n_X(X)} D(\mathsf{CBorn}_K) \tag{3.259}$$

which, on objects, sends $M^\bullet \mapsto (M^\bullet_n)_n$ where $M^\bullet_n = M^\bullet \widehat{\otimes}^{\mathbf{L}}_{\widehat{\mathcal{D}}_X(X)} \mathcal{D}^n_X(X)$.

Theorem 3.2.55. *The functor*

$$\phi : D_{\mathcal{C}}(X) \to \lim_n \mathrm{RMod}^{\mathrm{b,fg}}_{\mathcal{D}^n_X(X)} D(\mathsf{CBorn}_K) \tag{3.260}$$

is an equivalence of ∞-categories.

Proof. We will first show that, given a Cartesian section $(N^\bullet_n)_n$ belonging to the left-side of (3.260), then $N^\bullet := R\lim_n N^\bullet_n$ belongs to the full subcategory of $\mathcal{C}$-complexes, so that we obtain a right adjoint $\psi : (N^\bullet_n) \mapsto N^\bullet := R\lim_n N^\bullet_n$ to the functor ϕ above. (If we can show that $N^\bullet$ is a $\mathcal{C}$-complex, then it will immediately follow that ϕ is fully-faithful, as by definition we have $M^\bullet \simeq \psi\phi(M^\bullet)$ for any $\mathcal{C}$-complex $M^\bullet$).

For each $j \in \mathbf{Z}$, the system $\{H^j(N^\bullet_n)\}_n$ satisfies $H^j(N^\bullet_{n+1}) \widehat{\otimes}_{\mathcal{D}^{n+1}_X(X)} \mathcal{D}^n_X(X) \xrightarrow{\sim} H^j(N^\bullet_n)$, by flatness[27] of $\mathcal{D}^n_X(X)$ as a (left) $\mathcal{D}^{n+1}_X(X)$-module. In particular the system $\{H^j(N^\bullet_n)\}_n$ is pre-nuclear with dense images (c.f. the remark under [Bod21, Definition 5.24]). Hence the usual short-exact sequence

$$0 \to R^1 \lim_n H^{j-1}(N^\bullet_n) \to H^j(N^\bullet) \to \lim_n H^j(N^\bullet_n) \to 0 \tag{3.261}$$

together with the Mittag-Leffler result of [Bod21, Theorem 5.26] implies that $H^j(N^\bullet) \cong \lim_n H^j(N^\bullet_n)$ has coadmissible cohomology. We claim that for each m, the canonical morphism

$$N^\bullet \widehat{\otimes}^{\mathbf{L}}_{\widehat{\mathcal{D}}_X(X)} \mathcal{D}^m_X(X) \to N^\bullet_m \tag{3.262}$$

[26] To be precise, this abstract flatness together with [Bod21, Lemma 5.32] implies that finitely generated $\mathcal{D}^{n+1}_X(X)$-modules are acyclic for $-\widehat{\otimes}_{\mathcal{D}^{n+1}_X(X)} \mathcal{D}^n_X(X)$, and then an easy spectral-sequence argument gives the claim.

[27] See previous footnote.

is an equivalence. We may take cohomology. By [Bod21, Corollary 5.38] coadmissible right $\widehat{\mathcal{D}}_X(X)$-modules are acyclic for $(-)\widehat{\otimes}_{\widehat{\mathcal{D}}_X(X)}\mathcal{D}_X^m(X)$. This implies that

$$H^j(N^\bullet\widehat{\otimes}^{\mathbf{L}}_{\widehat{\mathcal{D}}_X(X)}\mathcal{D}_X^m(X)) \cong H^j(N^\bullet)\widehat{\otimes}_{\widehat{\mathcal{D}}_X(X)}\mathcal{D}_X^m(X) \tag{3.263}$$

for each $j \in \mathbf{Z}$: indeed, when $N^\bullet$ is bounded-above, this is an easy spectral-sequence argument, and in general one writes $N^\bullet$ as a (homotopy) colimit of its truncations, and uses that the derived tensor product commutes with colimits separately in each variable. However by properties of coadmissible modules we know that

$$H^j(N^\bullet)\widehat{\otimes}_{\widehat{\mathcal{D}}_X(X)}\mathcal{D}_X^m(X) \cong H^j(N_m). \tag{3.264}$$

This implies that (3.262) is an equivalence. By the comment above this implies that ϕ is fully-faithful. In fact, looking at (3.262) we see that $\phi\psi \simeq \mathrm{id}$, so that (ϕ,ψ) give an equivalence of categories. ☐

Remark 3.2.56. *We isolate the following useful fact from the proof of Theorem 3.2.55. Let $(N_n^\bullet)_n$ be a Cartesian section belonging to the right side of (3.260). Then for each $j \in \mathbf{Z}$, $N^\bullet := R\lim_n N_n^\bullet$ satisfies $H^j(N^\bullet) \xrightarrow{\sim} \lim_n H^j(N_n^\bullet)$.*

Lemma 3.2.57. *Let $M^\bullet \in D_{\mathcal{C}}(X)$ and let $n \geqslant 0$. Then for every $j \in \mathbf{Z}$ there is an isomorphism*

$$H^j(M^\bullet\widehat{\otimes}^{\mathbf{L}}_{\widehat{\mathcal{D}}_X(X)}\mathcal{D}_X^n(X)) \cong H^j(M^\bullet)\widehat{\otimes}_{\widehat{\mathcal{D}}_X(X)}\mathcal{D}_X^n(X). \tag{3.265}$$

Proof. By [Bod21, Corollary 5.38], coadmissible right $\widehat{\mathcal{D}}_X(X)$-modules are acyclic for $(-)\widehat{\otimes}^{\mathbf{L}}_{\widehat{\mathcal{D}}_X(X)}\mathcal{D}_X^n(X)$, which gives the claim (when $M^\bullet$ is bounded-above, this is an easy spectral-sequence argument, and in general one writes $M^\bullet$ as a homotopy colimit of its truncations and uses the commutation of tensor products with colimits). ☐

Lemma 3.2.58. *Let $M^\bullet \in D_{\mathcal{C}}(X)$ and let $U \subseteq X$ be an affinoid subdomain. Then:*

(i) $M_U^\bullet := M^\bullet\widehat{\otimes}^{\mathbf{L}}_{\widehat{\mathcal{D}}_X(X)}\widehat{\mathcal{D}}_X(U)$ belongs to $D_{\mathcal{C}}(U)$.

(ii) For each $j \in \mathbf{Z}$, one has $H^j(M_U^\bullet) \cong H^j(M^\bullet)\widehat{\otimes}_{\widehat{\mathcal{D}}_X(X)}\widehat{\mathcal{D}}_X(U)$.

In particular we obtain a sub-prestack

$$D_{\mathcal{C}}(-) \subseteq \mathrm{RMod}_{\widehat{\mathcal{D}}_X(-)}\, D(\mathsf{CBorn}_K) : X_w^{\mathsf{op}} \to \mathsf{Cat}_\infty. \tag{3.266}$$

Proof. In the following argument we always take n large enough so that U is p^n-accessible. By associativity properties of the tensor product and Lemma 3.2.52 we know that each $M_U^\bullet\widehat{\otimes}^{\mathbf{L}}_{\mathcal{D}_X(U)}\mathcal{D}_X^n(U)$ is bounded with finitely-generated cohomology. Now we show that the canonical morphism

$$M_U^\bullet \to R\lim_n M_U^\bullet\widehat{\otimes}^{\mathbf{L}}_{\mathcal{D}_X(U)}\mathcal{D}_X^n(U) \tag{3.267}$$

is an equivalence. Let us first examine the left side of (3.267). By [Bod21, Proposition 5.37], coadmissible right $\widehat{\mathcal{D}}_X(X)$-modules are acyclic for $(-)\widehat{\otimes}_{\widehat{\mathcal{D}}_X(X)}\widehat{\mathcal{D}}_X(U)$. This implies that

$$H^j(M_U^\bullet) \cong H^j(M^\bullet)\widehat{\otimes}_{\widehat{\mathcal{D}}_X(X)}\widehat{\mathcal{D}}_X(U) \tag{3.268}$$

for each $j \in \mathbf{Z}$. Indeed, when $M^\bullet$ is bounded above this follows from an easy spectral-sequence argument, and in general one writes $M^\bullet$ as a colimit of its truncations and uses the commutation of tensor products with colimits.

Now let us examine the right side of (3.267). Using Remark 3.2.56 above, one has

$$H^j(R\lim_n M_U^\bullet \widehat{\otimes}^{\mathbf{L}}_{\widehat{\mathcal{D}}_X(U)} \mathcal{D}_X^n(U)) \cong \lim_n H^j(M^\bullet \widehat{\otimes}^{\mathbf{L}}_{\widehat{\mathcal{D}}_X(X)} \mathcal{D}_X^n(U)). \tag{3.269}$$

By Lemma 3.2.57, we know that

$$H^j(M^\bullet \widehat{\otimes}^{\mathbf{L}}_{\widehat{\mathcal{D}}_X(X)} \mathcal{D}_X^n(U)) \cong H^j(M^\bullet) \widehat{\otimes}_{\widehat{\mathcal{D}}_X(X)} \mathcal{D}_X^n(U). \tag{3.270}$$

So, to show that (3.267) is an equivalence we are reduced to showing that

$$H^j(M^\bullet) \widehat{\otimes}_{\widehat{\mathcal{D}}_X(X)} \widehat{\mathcal{D}}_X(U) \to \lim_n H^j(M^\bullet) \widehat{\otimes}_{\widehat{\mathcal{D}}_X(X)} \mathcal{D}_X^n(U) \tag{3.271}$$

is an isomorphism for each $j \in \mathbf{Z}$. This follows from [Bod21, Proposition 5.33] (see also the preceding discussion about Ardakov–Wadsley's "cap tensor product" in *loc. cit.*). ☐

Corollary 3.2.59. *Suppose that $M^\bullet$ is a $\mathcal{C}$-complex. Then the underlying object $M^\bullet \in$ $\mathrm{QCoh}(X)$ belongs to $\mathrm{sFr}(X)$, so that there is an inclusion*

$$D_{\mathcal{C}}(X) \subseteq \mathrm{RMod}_{\widehat{\mathcal{D}}_X(X)} \mathrm{sFr}(X). \tag{3.272}$$

Proof. Let $U \subseteq X$ be an affinoid subdomain. Then by Lemma 3.2.58, $M^\bullet \widehat{\otimes}^{\mathbf{L}}_{\widehat{\mathcal{D}}_X(X)} \widehat{\mathcal{D}}_U(U)$ is again a $\mathcal{C}$-complex, whose underlying object in $\mathrm{QCoh}(X)$ is $M^\bullet \widehat{\otimes}^{\mathbf{L}}_A A_U$. Now the claim follows from Lemma 3.2.45. ☐

Lemma 3.2.58 also has the following consequence. By functoriality of pullbacks, we obtain a functor $X_w^{\mathsf{op}} \to \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)$ which sends $U \in X_w$ to the 1-morphism

$$\mathrm{RMod}_{\widehat{\mathcal{D}}_X(U)} D(\mathsf{CBorn}_K) \to \lim_n \mathrm{RMod}_{\mathcal{D}_X^n(U)} D(\mathsf{CBorn}_K). \tag{3.273}$$

The upshot of Lemma 3.2.58 is that, by restriction of this functor, we obtain a functor $X_w^{\mathsf{op}} \to \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)$ which sends $U \in X_w$ to the (invertible) 1-morphism

$$D_{\mathcal{C}}(U) \xrightarrow{\sim} \lim_n \mathrm{RMod}^{\mathrm{b,fg}}_{\mathcal{D}_X^n(U)} D(\mathsf{CBorn}_K). \tag{3.274}$$

We may use this implicitly in the following.

Theorem 3.2.60. *Let $\{U_i \to X\}_{i=1}^s$ be an admissible covering of X by affinoid subdomains. Let $Y := \coprod_{i=1}^s U_i \to X$. Then the canonical morphism*

$$D_{\mathcal{C}}(X) \to \lim_{[m]\in\Delta} D_{\mathcal{C}}(Y^{m+1/X}) \tag{3.275}$$

is an equivalence of ∞-categories.

Proof. In the following argument we always take n large enough so that all the $\{U_i\}_{i=1}^s$ are p^n-accessible. By Lemma 3.2.58 the following square commutes:

$$\begin{array}{ccc} \mathrm{RMod}^{\mathrm{b,fg}}_{\mathcal{D}_X^n(X)} D(\mathsf{CBorn}_K) & \longrightarrow & \mathrm{RMod}^{\mathrm{b,fg}}_{\mathcal{D}_X^n(Y^{m+1/X})} D(\mathsf{CBorn}_K) \\ \uparrow & & \uparrow \\ D_{\mathcal{C}}(X) & \longrightarrow & D_{\mathcal{C}}(Y^{m+1/X}) \end{array} \tag{3.276}$$

The bottom arrow is $-\widehat{\otimes}^{\mathbf{L}}_{\widehat{\mathcal{D}}_X(X)}\widehat{\mathcal{D}}_X(Y^{m+1/X})$, the right is $-\widehat{\otimes}^{\mathbf{L}}_{\widehat{\mathcal{D}}_X(Y^{m+1/X})}\mathcal{D}^n_X(Y^{m+1/X})$, the top is $-\widehat{\otimes}^{\mathbf{L}}_{\mathcal{D}^n_X(X)}\mathcal{D}^n_X(Y^{m+1/X})$, and the left is $-\widehat{\otimes}^{\mathbf{L}}_{\widehat{\mathcal{D}}_X(X)}\mathcal{D}^n_X(X)$. Passing to the limit over n, using Theorem 3.2.55, and then taking the limit over $[m] \in \Delta$, we see that the morphism (3.275) is equivalent to

$$\begin{aligned}\lim_n \mathrm{RMod}^{\mathrm{b,fg}}_{\mathcal{D}^n_X(X)} D(\mathsf{CBorn}_K) &\to \lim_{[m]\in\Delta}\lim_n \mathrm{RMod}^{\mathrm{b,fg}}_{\mathcal{D}^n_X(Y^{m+1/X})} D(\mathsf{CBorn}_K)\\ &\simeq \lim_n \lim_{[m]\in\Delta} \mathrm{RMod}^{\mathrm{b,fg}}_{\mathcal{D}^n_X(Y^{m+1/X})} D(\mathsf{CBorn}_K),\end{aligned} \tag{3.277}$$

where in the last line we used that limits commute with limits. Hence, the claim follows by taking limits over n in Theorem 3.2.53. □

Definition 3.2.61. *We define a pair of full subcategories*

$$(D^{\leqslant 0}_{\mathcal{C}}(X), D^{\geqslant 0}_{\mathcal{C}}(X)) \tag{3.278}$$

of $D_{\mathcal{C}}(X)$ by $M^\bullet \in D^{\leqslant 0}_{\mathcal{C}}(X)$ (resp. $M^\bullet \in D^{\geqslant 0}_{\mathcal{C}}(X)$) if $H^j(M^\bullet) = 0$ for all $j \geqslant 1$ (resp. if $H^j(M^\bullet) = 0$ for all $j \leqslant -1$).

Lemma 3.2.62. *With notations as above.*

(i) The pair $(D^{\leqslant 0}_{\mathcal{C}}(X), D^{\geqslant 0}_{\mathcal{C}}(X))$ determines a t-structure on $D_{\mathcal{C}}(X)$.

(ii) For $U \subseteq X$ an affinoid subdomain, the pullback $-\widehat{\otimes}^{\mathbf{L}}_{\widehat{\mathcal{D}}_X(X)}\widehat{\mathcal{D}}_X(U)$ restricts to a t-exact functor $D_{\mathcal{C}}(X) \to D_{\mathcal{C}}(U)$.

Proof. (i): We need to check that if $M^\bullet \in D_{\mathcal{C}}(X)$, then so does $\tau^{\leqslant 0}M^\bullet$ and $\tau^{\geqslant 0}M^\bullet$. Looking at Definition 3.2.43, we see that this follows immediately from Lemma 3.2.57 above.

(ii): This is Lemma 3.2.58. □

Lemma 3.2.63. *Let $[c, d] \subseteq \mathbf{R}$ be any interval with $d < \infty$. Let $\{U_i \to X\}^n_{i=1}$ be an admissible covering of X by affinoid subdomains. Let $Y = \coprod^n_{i=1} U_i$. Then the natural morphism*

$$D^{[c,d]}_{\mathcal{C}}(X) \to \lim_{[m]\in\Delta} D^{[c,d]}_{\mathcal{C}}(Y^{m+1/X}) \tag{3.279}$$

is an equivalence of ∞-categories.

Proof. Let $(M^\bullet_m)_{[m]\in\Delta}$ be an object of the right-side of (3.279). Due to Theorem 3.2.60, and Lemma 3.2.62 the only thing to prove is that $M^\bullet_{-1} := R\lim_{[m]\in\Delta} M^\bullet_m$ has cohomology in the interval $[c, d]$. One argues in essentially the same way as the proof of Theorem 3.2.53. We consider the hypercohomology spectral sequence

$$E^{pq}_2 : R^p \lim_{[m]\in\Delta} H^q M^\bullet_m \Rightarrow H^{p+q}(M^\bullet_{-1}). \tag{3.280}$$

This converges because $(M^\bullet_m)_{[m]\in\Delta}$ is uniformly homologically bounded by Lemma 3.2.58. Lemma 3.2.58 also implies that

$$H^q(M^\bullet_k)\widehat{\otimes}_{\widehat{\mathcal{D}}_X(Y^{k+1/X})}\widehat{\mathcal{D}}_X(Y^{l+1/X}) \cong H^q(M^\bullet_l) \tag{3.281}$$

is an isomorphism for every cosimplicial morphism $[l] \to [k]$. Then the counterpart of Kiehl's theorem in this setting [AW19, §8] implies that

$$R^p \lim_{[m]\in\Delta} H^q M_n^\bullet = 0, \text{ whenever } p > 0. \tag{3.282}$$

Thus the spectral-sequence (3.280) collapses and gives an isomorphism

$$H^n M_{-1}^\bullet \cong \lim_{[m]\in\Delta} H^n M_m^\bullet = \mathrm{eq}(H^n(M_0^\bullet) \rightrightarrows H^n(M_1^\bullet)), \tag{3.283}$$

which shows that the cohomological amplitude of $M_{-1}^\bullet$ is contained in the same interval as $M_0^\bullet$. ☐

3.2.8 Compatibility with restrictions

Let $X = \mathrm{Sp}(A)$ be a classical affinoid equipped with an étale morphism $X \to \mathbf{D}_K^r$. Let $U \subseteq X$ be an affinoid subdomain. We recall from Example 3.2.10 that the square

$$\begin{array}{ccc} U & \longrightarrow & U_{\mathrm{str}} \\ \downarrow & \ulcorner & \downarrow \\ X & \longrightarrow & X_{\mathrm{str}} \end{array} \tag{3.284}$$

is Cartesian in the category PStk. We recall that $\mathrm{Strat}(-)$ and $\mathrm{QCoh}(-)$ are functorial with respect to upper-star pullbacks. Let X_w be the poset of affinoid subdomains of X. Using the base-change equivalences from the six-functor formalism obtained in Theorem 3.1.69 we obtain a natural transformation of functors $X_w^{\mathsf{op}} \to \mathsf{Cat}_\infty$:

$$p_{(-),!} : \mathrm{QCoh}(-) \to \mathrm{Strat}(-), \tag{3.285}$$

whose component for each $U \in X_w$ is the lower-shriek pushforwards $p_{U,!} : \mathrm{QCoh}(U) \to \mathrm{Strat}(U)$ induced by $p_U : U \to U_{\mathrm{str}}$. By passing to right adjoints pointwise we a *lax natural transformation*:

$$p_{(-)}^! : \mathrm{Strat}(-) \to \mathrm{QCoh}(-), \tag{3.286}$$

whose component over $U \in X_w$ is the upper-shriek pullback $p_U^!$. This essentially means that the natural transformations in each square

$$\begin{array}{ccc} \mathrm{Strat}(X) & \xrightarrow{p_X^!} & \mathrm{QCoh}(X) \\ \downarrow{\scriptstyle t_{\mathrm{str}}^*} & \swarrow & \downarrow{\scriptstyle t^*} \\ \mathrm{Strat}(U) & \xrightarrow{p_U^!} & \mathrm{QCoh}(U) \end{array} \tag{3.287}$$

can be composed vertically in a natural way. Here $t : U \hookrightarrow X$ is the inclusion.

Lemma 3.2.64. *With notations as above. The natural transformation*

$$t^* p_X^! \to p_U^! t_{\mathrm{str}}^* \tag{3.288}$$

restricts to an equivalence on the full subcategory spanned by those $M \in \mathrm{Strat}(X)$ such that $p_X^! M$ belongs to $\mathrm{Fr}(\mathrm{X}) \subseteq \mathrm{QCoh}(X)$.

Before proving this Lemma, let me say why it is relevant, first introducing the following Definition:

Definition 3.2.65. *We define* $\mathrm{Strat}_{\mathrm{Fr}}(X) \subseteq \mathrm{Strat}(X)$ *(resp.* $\mathrm{Strat}_{\mathrm{sFr}}(X) \subseteq \mathrm{Strat}(X)$*) to be the full sub-*$\infty$*-category spanned by objects* M *such that* $p_X^! M$ *belongs to* $\mathrm{Fr}(X)$ *(resp.* $\mathrm{sFr}(X)$*).*

The upshot of Lemma 3.2.64 is twofold:

Corollary 3.2.66. *(i)* $\mathrm{Strat}_{\mathrm{sFr}}(-)$ *forms a sub-prestack of* $\mathrm{Strat}(-)$ *on* X_w*, with respect to the upper-star functors.*

(ii) The lax natural transformation (3.286) *restricts to a natural transformation*

$$p^!_{(-)} : \mathrm{Strat}_{\mathrm{sFr}}(-) \to \mathrm{sFr}(-). \tag{3.289}$$

We emphasise that the lax structure here is in fact strong, so that (3.289) *is a natural transformation of functors* $X_w^{\mathsf{op}} \to \mathsf{Cat}_\infty$*, in which both sides are viewed as prestacks via the* upper-star *functors.*

Proof of Lemma 3.2.64. We would like to show that, for $M \in \mathrm{Strat}_{\mathrm{Fr}}(X)$, and an affinoid subdomain $t : U \hookrightarrow X$, the natural morphism

$$t^* p_X^! M \to p_U^! t^*_{\mathrm{str}} M \tag{3.290}$$

is an equivalence. Let us first consider the case when $M = p_{X,!}N$ for some $N \in \mathrm{sFr}(X)$. (That such M belongs to $\mathrm{Strat}_{\mathrm{sFr}}(X)$ is a consequence of Proposition 3.2.41). Then, we are asking for

$$t^*\mathcal{D}_X^\infty N = t^* p_X^! p_{X,!} N \to p_U^! t^*_{\mathrm{str}} p_{X,!} N \simeq p_U^! p_{U,!} t^* N = \mathcal{D}_U^\infty t^* N \tag{3.291}$$

to be an equivalence, where in the last part we used base-change. Because N and t^*N are a Frechet-strict complexes, by Theorem 3.2.40 this will follow if $t^*\mathcal{D}_X^\infty 1_X \simeq \mathcal{D}_U^\infty 1_U$. But this can be deduced, for instance, from Proposition 3.2.39, because $\widehat{\mathcal{D}}_X(X) \widehat{\otimes}^{\mathbf{L}}_A A_U \xrightarrow{\sim} \widehat{\mathcal{D}}_U(U)$.

Now let us consider the general case. By monadicity (Theorem 3.2.34) of the adjunction $p_{X,!} \dashv p_X^!$, every $M \in \mathrm{Strat}_{\mathrm{Fr}}(X)$ may be expressed as the colimit of a $p_X^!$-split simplicial object:

$$M \simeq \operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} p_{X,!} N_n, \tag{3.292}$$

where $N_n := (p_X^! p_{X,!})^n p_X^! M$. Each N_n belongs to $\mathrm{sFr}(X)$, by Proposition 3.2.41. Using that $p_X^!$ (resp. $p_U^!$) commutes with geometric realizations of $p_X^!$-split (resp. $p_U^!$-split) simplicial objects, and that the upper-star functors are colimit-preserving, we compute

$$\begin{aligned} t^* p_X^! M &\simeq \operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} t^* p_X^! p_{X,!} N_n \\ &\simeq \operatorname*{colim}_{[n]\in\Delta^{\mathsf{op}}} p_U^! t^*_{\mathrm{str}} p_{X,!} N_n \simeq p_U^! t^*_{\mathrm{str}} M, \end{aligned} \tag{3.293}$$

where we used the previous case. □

Lemma 3.2.67. *With notations as above. Let* $t : U \to X$ *be the inclusion.*

(i) There are restriction functors

$$\mathrm{Mod}_{\mathcal{D}_X^\infty} \mathrm{sFr}(X) \to \mathrm{Mod}_{\mathcal{D}_U^\infty} \mathrm{sFr}(U) \tag{3.294}$$

induced by t^* *on* $\mathrm{sFr}(X)$*. These functors are natural in* U*, so that we obtain a prestack* $\mathrm{Mod}_{\mathcal{D}_{(-)}^\infty} \mathrm{sFr}(-)$ *on* X_w*.*

(ii) The equivalences $\mathrm{Strat}_{\mathrm{sFr}}(X) \simeq \mathrm{Mod}_{\mathcal{D}_X^\infty} \mathrm{sFr}(X)$ *is compatible with these restriction functors, and the upper-star pullback functors on* $\mathrm{Strat}_{\mathrm{sFr}}$ *(c.f. Corollary 3.2.66), so that we obtain an equivalence*

$$\mathrm{Mod}_{\mathcal{D}_{(-)}^\infty} \mathrm{sFr}(-) \simeq \mathrm{Strat}_{\mathrm{sFr}}(-) \tag{3.295}$$

of prestacks on X_w.

Proof. This follows by combining Corollary 3.2.66, Theorem 3.2.34 and Corollary 2.2.7. □

Remark 3.2.68. *Before moving on we give a more concrete description of the restriction functors in (i). Due to Lemma 3.2.64 and base-change, for each* $M \in \mathrm{sFr}(X)$ *there is a canonical equivalence*

$$\sigma : \mathcal{D}_U^\infty t^* \simeq t^* \mathcal{D}_X^\infty \tag{3.296}$$

of functors on $\mathrm{sFr}(X)$. *Due to naturality of base-change, this satisfies the expected naturalities making* t^* *into a* monad functor *from* $\mathcal{D}_U^\infty 1_U$ *to* $\mathcal{D}_X^\infty 1_X$. *Roughly speaking, this says that there are diagrams*

$$\begin{array}{ccc} t^* & \xrightarrow{\eta_U t^*} & \mathcal{D}_X^\infty t^* \\ & \searrow^{t^*\eta_X} & \downarrow \sigma \\ & & t^*\mathcal{D}_U^\infty \end{array} \tag{3.297}$$

and

$$\begin{array}{ccccc} \mathcal{D}_X^\infty t^* \mathcal{D}_U^\infty & \xrightarrow{\mathcal{D}_X^\infty \sigma} & \mathcal{D}_X^\infty \mathcal{D}_X^\infty t^* & \xrightarrow{\mu_X t^*} & \mathcal{D}_X^\infty t^* \\ \uparrow \sigma\mathcal{D}_U^\infty & & & & \uparrow \sigma \\ t^*\mathcal{D}_U^\infty \mathcal{D}_U^\infty & & \xrightarrow{t^*\mu_U} & & t^*\mathcal{D}_U^\infty \end{array} \tag{3.298}$$

of functors on $\mathrm{sFr}(X)$*, which are homotopy-commutative, together with various higher coherences. Then, if* $M \in \mathrm{Mod}_{\mathcal{D}_X^\infty} \mathrm{sFr}(X)$*, there is a canonical* $\mathcal{D}_U^\infty$*-module structure on* t^*M *in which the action of* $\mathcal{D}_U^\infty$ *is given by the composite*

$$\mathcal{D}_U^\infty t^* M \xrightarrow{\sigma} t^* \mathcal{D}_X^\infty M \xrightarrow{t^* \mathrm{act}} t^* M. \tag{3.299}$$

Lemma 3.2.69. *(i) There are restriction functors*

$$\mathrm{RMod}_{\mathcal{D}_X^\infty 1_X} \mathrm{QCoh}(X) \to \mathrm{RMod}_{\mathcal{D}_U^\infty 1_U} \mathrm{QCoh}(U) \tag{3.300}$$

induced by t^* *on* $\mathrm{sFr}(X)$*. These functors are natural in* U*, so that we obtain a prestack on* X_w.

(ii) The equivalence $\mathrm{RMod}_{\mathcal{D}_X^\infty 1_X} \mathrm{sFr}(X) \simeq \mathrm{Mod}_{\mathcal{D}_X^\infty} \mathrm{sFr}(X)$ *is compatible with these restrictions and those coming from (i), so that we obtain an equivalence*

$$\mathrm{RMod}_{\mathcal{D}_{(-)}^\infty 1_{(-)}} \mathrm{sFr}(-) \simeq \mathrm{Mod}_{\mathcal{D}_{(-)}^\infty} \mathrm{sFr}(-) \tag{3.301}$$

of prestacks on X_w.

Proof. (i): We note that there is a canonical equivalence of A-A_U bimodule objects

$$\mathcal{D}_U^\infty 1_U \simeq \mathcal{D}_X^\infty 1_X \widehat{\otimes}_A^{\mathbf{L}} A_U. \tag{3.302}$$

Thus for $M \in \mathrm{RMod}_{\mathcal{D}_X^\infty 1_X} \mathrm{QCoh}(X)$, the object $t^*M = M\widehat{\otimes}_A^{\mathbf{L}} A_U$ obtains the canonical stucture of a $\mathcal{D}_U^\infty 1_U$-module object via

$$(M\widehat{\otimes}_A^{\mathbf{L}} A_U)\widehat{\otimes}_{A_U}^{\mathbf{L}} \mathcal{D}_U^\infty 1_U \simeq (M\widehat{\otimes}_A^{\mathbf{L}} \mathcal{D}_X^\infty 1_X)\widehat{\otimes}_A^{\mathbf{L}} A_U \to M\widehat{\otimes}_A^{\mathbf{L}} A_U, \tag{3.303}$$

which gives the required restriction functors. A different way to say this is that the functor $t^* = (-)\widehat{\otimes}_A^{\mathbf{L}} A_U$, together with the equivalence

$$\sigma' : (-)\widehat{\otimes}_A^{\mathbf{L}} A_U \widehat{\otimes}_{A_U}^{\mathbf{L}} \mathcal{D}_U^\infty 1_U \simeq (-)\widehat{\otimes}_A^{\mathbf{L}} \mathcal{D}_X^\infty 1_X \widehat{\otimes}_A^{\mathbf{L}} A_U \tag{3.304}$$

gives a monad functor (in the sense of Remark 3.2.68) from $(-)\widehat{\otimes}_{A_U}^{\mathbf{L}} \mathcal{D}_U^\infty 1_U$ to $(-)\widehat{\otimes}_A^{\mathbf{L}} \mathcal{D}_X^\infty 1_X$.

(ii): We need to show that the natural transformation (3.210) is compatible with the monad functors σ and σ' of (3.296) and (3.304) respectively. Applying Lemma 2.2.21 (with $\mathscr{V} = D(\mathsf{CBorn}_K)$ and $\mathcal{M} = D(\mathsf{CBorn}_K)$ and, in the notations of that Lemma), there is an adjunction

$$\iota : {}_A\mathrm{BMod}_B\, \mathscr{V} \leftrightarrows \mathrm{Fun}_{\mathscr{V}}^{\mathrm{lax}}(\mathrm{Mod}_A \mathscr{V}, \mathrm{Mod}_B \mathscr{V}) : \kappa. \tag{3.305}$$

in which the left adjoint ι is fully-faithful. The equivalence $\sigma : \mathcal{D}_U^\infty t^* \simeq t^*\mathcal{D}_X^\infty$ of functors on $\mathrm{sFr}(X)$ comes from restricting the natural transformation $\tau : t^*\mathcal{D}_X^\infty \to \mathcal{D}_U^\infty t^*$ of functors on $\mathrm{QCoh}(X) = \mathrm{Mod}_A \mathscr{V}$. We may view τ as a morphism in the category on the the right side of (3.305). Now we note that the inverse of σ' agrees with (the restriction to $\mathrm{sFr}(X)$ of) $\iota\kappa(\tau)$: indeed, $\kappa\iota\kappa(\tau)$ identifies with $\mathcal{D}_X^\infty 1_X \widehat{\otimes}_A^{\mathbf{L}} A_U \xrightarrow{\sim} \mathcal{D}_U^\infty 1_U$. Hence, using that $\iota\kappa \to \mathrm{id}$ is a natural transformation we obtain the desired commutative square

$$\begin{array}{ccc} \mathcal{D}_U^\infty(-\widehat{\otimes}_A^{\mathbf{L}} A_U) & \xrightarrow{\simeq\sigma} & (\mathcal{D}_X^\infty(-))\widehat{\otimes}_A^{\mathbf{L}} A_U \\ \uparrow & & \uparrow \\ (-)\widehat{\otimes}_A^{\mathbf{L}} A_U \widehat{\otimes}_{A_U}^{\mathbf{L}} \mathcal{D}_U^\infty 1_U & \xrightarrow{\simeq\sigma'} & (-)\widehat{\otimes}_A^{\mathbf{L}} \mathcal{D}_X^\infty 1_X \widehat{\otimes}_A^{\mathbf{L}} A_U \end{array} \tag{3.306}$$

of endofunctors on $\mathrm{sFr}(X)$ in which the vertical arrows are induced by (3.210). □

It is not hard to see that the equivalence

$$\mathrm{RMod}_{\mathcal{D}_X^\infty 1_X} \mathrm{QCoh}(X) \simeq \mathrm{RMod}_{\widehat{\mathcal{D}}_X(X)} D(\mathsf{CBorn}_K) \tag{3.307}$$

is compatible with restrictions to $U \in X_w$, so that there is an equivalence of prestacks

$$\mathrm{RMod}_{\mathcal{D}_{(-)}^\infty 1_{(-)}} \mathrm{QCoh}(-) \simeq \mathrm{RMod}_{\widehat{\mathcal{D}}_{(-)}(-)} D(\mathsf{CBorn}_K) \tag{3.308}$$

on X_w. We recall from Definition 3.2.43 that the ∞-category $D_{\mathcal{C}}(X)$ is defined as a certain full subcategory of $\mathrm{RMod}_{\widehat{\mathcal{D}}_X(X)} D(\mathsf{CBorn}_K)$. Under the equivalence (3.307), the image of $D_{\mathcal{C}}(X)$ factors through the full subcategory $\mathrm{RMod}_{\mathcal{D}_X^\infty 1_X} \mathrm{sFr}(X)$, by Corollary 3.2.59. In particular we obtain a fully-faithful functor

$$D_{\mathcal{C}}(X) \hookrightarrow \mathrm{RMod}_{\mathcal{D}_X^\infty 1_X} \mathrm{sFr}(X) \tag{3.309}$$

which is compatible with restrictions. Now combining all the assertions of Lemma 3.2.67 we obtain the following.

Theorem 3.2.70. *The functor $D_{\mathcal{C}}(X) \to \mathrm{Strat}(X)$ is compatible with the restrictions induced by the upper-star pullback functors on* Strat*. That is, there is a morphism*

$$D_{\mathcal{C}}(-) \to \mathrm{Strat}(-) \tag{3.310}$$

of prestacks on X_w, which is pointwise fully-faithful.

3.2.9 Globalising the embedding of $\mathcal{C}$-complexes

In this subsection X now denotes an arbitrary smooth rigid analytic variety. Let $X_w(\mathcal{T})$ denote the poset of affinoid subdomains of X which are étale over a polydisk (this is a basis for the weak topology). There is obviously a functor

$$X_w(\mathcal{T}) \to X_{\mathrm{strong}} \tag{3.311}$$

where the latter is the poset of all admissible open subsets of X equipped with the strong G-topology.

Definition 3.2.71. *The stack of $\mathcal{C}$-complexes on X is defined to be the right Kan extension of the functor $D_{\mathcal{C}}(-) : X_w^{\mathsf{op}}(\mathcal{T}) \to \mathsf{Cat}_\infty$ along $X_w^{\mathsf{op}}(\mathcal{T}) \to X_{\mathrm{strong}}$.*

With this definition, one has (for an arbitrary rigid smooth variety X):

$$D_{\mathcal{C}}(X) = \varprojlim_U D_{\mathcal{C}}(U), \tag{3.312}$$

where the limit runs over all affinoid subdomains $U \subseteq X$ which are étale over a polydisk.

Theorem 3.2.72. *With notations as above:*

(i) $D_{\mathcal{C}}(-)$ is a sheaf of ∞-categories on X_{strong}.

(ii) There is a fully-faithful functor $D_{\mathcal{C}}(X) \hookrightarrow \mathrm{Strat}(X)$, which is compatible with restrictions to admissible open subsets $U \subseteq X$, so that we obtain a morphism

$$D_{\mathcal{C}}(-) \to \mathrm{Strat}(-) \tag{3.313}$$

of Cat_∞-valued sheaves on X_{strong} which is pointwise fully-faithful.

Proof. (i): Because $X_w(\mathcal{T})$ is a basis for X_{strong}, this follows from [Man22, Proposition A.3.11(ii)] and Theorem 3.2.55.

(ii): By Lemma 3.2.17, we know that for a smooth (classical) rigid variety X, that

$$\mathrm{Strat}(X) \simeq \varprojlim_U \mathrm{Strat}(U), \tag{3.314}$$

where the limit runs over all affinoid subdomains which are étale over a polydisk. In particular the morphism (3.313) may be constructed using Theorem 3.2.70 and Kan extension. The fully-faithfulness also follows from Theorem 3.2.70, together with the fact that the mapping space in a limit of ∞-categories, is the limit of the mapping spaces. □

By Lemma 3.2.63, we obtain a subsheaf $D_{\mathcal{C}}^{\heartsuit}(-)$ of $D_{\mathcal{C}}(-)$ on $X_w(\mathcal{T})$. By Kan extension, we obtain a subsheaf $D_{\mathcal{C}}^{\heartsuit}(-)$ of $D_{\mathcal{C}}(-)$ on X_{strong}.

Theorem 3.2.73. *Let X be a smooth classical rigid-analytic space. The category $D_{\mathcal{C}}^{\heartsuit}(X)$ is equivalent to the category of coadmissible $\widehat{\mathcal{D}}_X$-modules of [AW19, §9.4].*

Proof. By construction, the abelian category $D_{\mathcal{C}}(X)^{\heartsuit} = D_{\mathcal{C}}^{\heartsuit}(X)$ satisfies

$$D_{\mathcal{C}}^{\heartsuit}(X) \simeq \varprojlim_{U \subseteq X} D_{\mathcal{C}}^{\heartsuit}(U), \tag{3.315}$$

where the limit runs over all affinoid subdomains $U \subseteq X$ which are étale over a polydisk. It is clear that each $D_{\mathcal{C}}^{\heartsuit}(U)$ identifies with the category of coadmissible $\widehat{\mathcal{D}}_U(U)$-modules. The limit on the left is the (2,1)-limit in the sense of ordinary category theory. To be precise, the $(2,1)$-limit is defined to be the category of Cartesian sections of the Grothendieck fibration corresponding to the Cat-valued presheaf $D_{\mathcal{C}}^{\heartsuit}(-)$. In concrete terms, its objects are collections $(M_U)_U$ of objects in each category equipped with the data of equivalences $\phi_{UV} : M_U \widehat{\otimes}_{\widehat{\mathcal{D}}_U(U)} \widehat{\mathcal{D}}_V(V) \xrightarrow{\sim} M_V$ for every affinoid subdomain $V \subseteq U$, which satisfy an obvious cocycle condition. Thus, using [AW19, §8], we are reduced to proving that

$$\{\text{coadmissible } \mathcal{D}_X\text{-modules}\} \simeq \varprojlim_{U} \{\text{coadmissible } \mathcal{D}_U\text{-modules}\} \tag{3.316}$$

where the limit on the right is the $(2,1)$-limit. Given [AW19, Theorem 9.4], this is tautological: the functor from left to right is given by restriction and the functor from right to left glues a sheaf of $\mathcal{D}_X$-modules from local data, and the equivalence (3.316) expresses the fact that being coadmissible is local on $X_w(\mathcal{T})$. □

Corollary 3.2.74. *Let X be a smooth rigid-analytic variety. There is a fully-faithful functor*

$$\{\textit{coadmissible } \mathcal{D}_X\textit{-modules}\} \hookrightarrow \mathrm{Strat}(X). \tag{3.317}$$

In the future we may investigate the essential image of the functor $D_{\mathcal{C}}(X) \hookrightarrow \mathrm{Strat}(X)$, for X a smooth rigid-analytic space. As the next Proposition shows, it is not completely obvious how to do this.

Proposition 3.2.75. *Let $X = \mathrm{dSp}(K\langle x\rangle)$ be the closed unit disk. Then the image of $\widehat{\mathcal{D}}_X(X)$ under the functor $D_{\mathcal{C}}(X) \to \mathrm{Strat}(X)$ is not a dualizable*[28] *object.*

Proof. Set $A := K\langle x\rangle$. Recall the canonical morphism $p : X \to X_{\mathrm{str}}$. The image of $\widehat{\mathcal{D}}_X(X)$ under the functor $D_{\mathcal{C}}(X) \to \mathrm{Strat}(X)$ is the object $p_* 1_X \in \mathrm{Strat}(X) = \mathrm{QCoh}(X_{\mathrm{str}})$. Suppose for a contradiction that this object is dualizable. The functor p^* is symmetric monoidal, hence it preserves dualizable objects. So

$$p^* p_* 1_X \simeq (A \widehat{\otimes}_K A)^{\dagger}_{\Delta} \simeq A \widehat{\otimes}_K K\langle t/p^{\infty}\rangle \tag{3.318}$$

would be a dualizable object of[29] $\mathrm{QCoh}(X)$. We claim it isn't: after simplification using associativity of $\widehat{\otimes}^{\mathbf{L}}$, this reduces to showing that the canonical morphism

$$R\underline{\mathrm{Hom}}_K(K\langle u/p^{\infty}\rangle, A) \widehat{\otimes}^{\mathbf{L}}_K K\langle t/p^{\infty}\rangle \to R\underline{\mathrm{Hom}}_K(K\langle u/p^{\infty}\rangle, A \widehat{\otimes}_K K\langle t/p^{\infty}\rangle), \tag{3.319}$$

is not an equivalence. Taking zeroth cohomology[30], and then arguing using cofinality, it is sufficient to show that

$$\operatorname*{colim}_m \lim_n A \widehat{\otimes}_K K\langle p^n s, t/p^m\rangle \to \lim_n \operatorname*{colim}_m A \widehat{\otimes}_K K\langle p^n s, t/p^m\rangle, \tag{3.320}$$

[28] We recall that an object A of a symmetric monoidal ∞-category $(\mathscr{V}, \otimes)$ is called *dualizable* if there exists another object $A^{\vee} \in \mathscr{V}$ such that $A \otimes (-)$ is adjoint to $A^{\vee} \otimes (-)$.

[29] Here $A \widehat{\otimes}_K K\langle t/p^{\infty}\rangle$ is viewed as an A-module by the action on A only.

[30] In fact one can show that both sides are concentrated in degree 0, but this is not necessary for the argument.

is not an equivalence; here s is dual to u. We can explicitly describe both sides of (3.320). The left side is:

$$\left\{ \sum_{k,l} c_{kl} s^k t^l : c_{kl} \in A, \exists m \forall n \ \|c_{kl}\| p^{nk-ml} \to 0 \text{ as } k,l \to \infty \right\}, \tag{3.321}$$

whereas the right side has the order of quantifiers reversed:

$$\left\{ \sum_{k,l} c_{kl} s^k t^l : c_{kl} \in A, \forall n \exists m \ \|c_{kl}\| p^{nk-ml} \to 0 \text{ as } k,l \to \infty \right\}, \tag{3.322}$$

and so we can exhibit an element which belongs to (3.322) but not (3.321), for instance $\sum_{k,l} s^k t^l$. □

Chapter 4

A possible alternative via algebraic theories

This chapter is independent of the rest of the thesis and is not used elsewhere.

4.1 On six-functor formalisms associated to Lawvere theories

Let Fin denote the category of finite sets.

Definition 4.1.1. *A* Lawvere theory *is a small category $\mathscr{T}$ with finite products, equipped with a finite-product preserving functor $\mathscr{T} \to \mathsf{Fin}$ which is the identity on objects. We write $\mathbf{A}^1_{\mathscr{T}} \in \mathscr{T}$ for the object corresponding to the singleton $* \in \mathsf{Fin}$.*

A morphism of Lawvere theories *is a finite product-preserving functor over Fin. We let Law denote the category of Lawvere theories.*

Definition 4.1.2. *Let Λ be a small set. A* Λ-sorted Lawvere theory *is a small category $\mathscr{T}$ with finite products, equipped with a finite-product preserving functor $\mathscr{T} \to \mathsf{Fin}_{/\Lambda}$. We write $\mathbf{D}^1_{\mathscr{T}}(\lambda) \in \mathscr{T}$ for the object corresponding to $\lambda : * \to \Lambda$. A morphism of Λ-sorted Lawvere theories is a finite product-preserving functor over $\mathsf{Fin}_{/\Lambda}$.*

Definition 4.1.3. *Let $\mathscr{T}$ be a (possibly Λ-sorted) Lawvere theory.*

(i) We define the category of algebras of $\mathscr{T}$ as

$$\mathsf{Alg}_{\mathscr{T}} := \mathrm{Fun}^{\Pi}(\mathscr{T}, \mathsf{Set}), \tag{4.1}$$

where Fun^{Π} denotes the finite-product preserving functors.

(ii) We define the ∞-category of derived algebras of $\mathscr{T}$ as

$$\mathsf{dAlg}_{\mathscr{T}} := \mathrm{Fun}^{\Pi}(\mathscr{T}, \infty\mathsf{Grpd}) = \mathrm{sInd}(\mathscr{T}^{\mathsf{op}}). \tag{4.2}$$

The category $\mathsf{Alg}_{\mathscr{T}}$ is equivalent to the full subcategory of $\mathsf{dAlg}_{\mathscr{T}}$ on discrete (i.e, 0-truncated) objects, and this inclusion is reflective, with the reflector induced by $\pi_0 : \infty\mathsf{Grpd} \to \mathsf{Set}$.

Example 4.1.4. *The full subcategory of $\mathsf{Rings}^{\mathsf{op}}$ on the objects $\{\mathbf{A}^n_{\mathbf{Z}}\}_{n \geqslant 0}$ forms a Lawvere theory, denoted $\mathsf{Com}_{\mathbf{Z}}$. Its category of algebras is the ordinary category of rings and its ∞-category of derived algebras is the ∞-category of animated rings.*

Definition 4.1.5. *Let $\mathscr{V}$ be a presentably symmetric monoidal, stable ∞-category and let $\mathscr{T}$ be a Lawvere theory. A $\mathscr{V}$-*realization *of $\mathscr{T}$ is a finite coproduct-preserving functor $\mathscr{T}^{\mathsf{op}} \to \mathsf{CAlg}(\mathscr{V})$.*

Let $\mathscr{T}^{\mathsf{op}} \to \mathsf{CAlg}(\mathscr{V})$ be a $\mathscr{V}$-realization of a Lawvere theory $\mathscr{T}$. By [Lur09b, Proposition 5.5.8.15(3)] this extends uniquely to a colimit-preserving functor

$$\mathsf{dAlg}_{\mathscr{T}} \to \mathsf{CAlg}(\mathscr{V}). \tag{4.3}$$

If we once again set $\mathscr{E} := \mathsf{CAlg}(\mathscr{V})^{\mathsf{op}}$ then we obtain a limit-preserving functor $\mathsf{dAlg}_{\mathscr{T}}^{\mathsf{op}} \to \mathscr{E}$ and hence a symmetric monoidal functor

$$\mathrm{Corr}(\mathsf{dAlg}_{\mathscr{T}}^{\mathsf{op}}, \mathrm{all})^{\otimes} \to \mathrm{Corr}(\mathscr{E}, \mathrm{all})^{\otimes}. \tag{4.4}$$

By post-composing with the six-functor formalism of Proposition 2.3.13 we obtain a six-functor formalism

$$\mathrm{QCoh} : \mathrm{Corr}(\mathsf{dAlg}_{\mathscr{T}}^{\mathsf{op}}, \mathrm{all})^{\otimes} \to \mathsf{Pr}_{\mathsf{st}}^{L,\otimes}, \tag{4.5}$$

in which every morphism f in $\mathsf{dAlg}_{\mathscr{T}}^{\mathsf{op}}$ satisfies $f_! = f_*$. The Yoneda embedding induces a morphism of geometric setups $(\mathsf{dAlg}_{\mathscr{T}}^{\mathsf{op}}, \mathrm{all}) \to (\mathrm{PSh}(\mathsf{dAlg}_{\mathscr{T}}^{\mathsf{op}}), \mathrm{rep})$ and again by [Man22, Proposition A.5.16] we may extend QCoh to a six-functor formalism

$$\mathrm{QCoh} : \mathrm{Corr}(\mathrm{PSh}(\mathsf{dAlg}_{\mathscr{T}}^{\mathsf{op}}), \mathrm{rep})^{\otimes} \to \mathsf{Pr}_{\mathsf{st}}^{L,\otimes}. \tag{4.6}$$

In this six-functor formalism, for every morphism $g \in \mathrm{rep}$ satisfies $g_! \simeq g_*$ (the proof of this fact is identical to Corollary 2.3.16).

Theorem 4.1.6. *There exists a (minimal) class of edges $E \supseteq \mathrm{rep}$ of $\mathrm{PSh}(\mathsf{dAlg}_{\mathscr{T}}^{\mathsf{op}})$ such that* QCoh *extends to a six-functor formalism on $(\mathrm{PSh}(\mathsf{dAlg}_{\mathscr{T}}^{\mathsf{op}}), E)$, and E is stable under disjoint unions, $*$-local on the target, $!$-local on the source, is tame, and satisfies $E \subseteq \delta E$.*

Proof. This is very similar to the proof of Theorem 2.3.17 and so we omit it. ☐

Remark 4.1.7. *The above construction has an obvious generalization to sorted and infinitary Lawvere theories.*

Example 4.1.8 (Smooth geometry)**.** *Consider the (Archimedean) Banach field $(\mathbf{R}, |\cdot|_\infty)$ and the category $\mathsf{CBorn}_{\mathbf{R}}$ of (Archimedean) complete bornological $\mathbf{R}$-vector spaces [BK17]. We consider the Lawvere theory $\mathscr{T} = \mathsf{CartSm}$ in which*

$$\mathrm{Map}(m, n) := \mathrm{Hom}(C^\infty(\mathbf{R}^n), C^\infty(\mathbf{R}^m)). \tag{4.7}$$

Here Hom *denotes the continuous morphisms of Fréchet $\mathbf{R}$-algebras. The functor which endows $C^\infty(\mathbf{R}^n)$ with its precompact bornology [BK17] determines a finite coproduct-preserving, fully-faithful functor*

$$\mathsf{CartSm}^{\mathsf{op}} \to \mathsf{CAlg}(D_{\geqslant 0}(\mathsf{CBorn}_{\mathbf{R}})) \subseteq \mathsf{CAlg}(D(\mathsf{CBorn}_{\mathbf{R}})). \tag{4.8}$$

Therefore we may apply the above formalism with $\mathscr{V} = D(\mathsf{CBorn}_{\mathbf{R}})$. The category of algebras for this Lawvere theory is $\mathsf{C}^\infty\mathsf{Alg}$ and the ∞-category of derived algebras is the ∞-category of derived C^∞-algebras:

$$\mathsf{dC}^\infty\mathsf{Alg} := \mathrm{Fun}^{\Pi}(\mathsf{CartSm}, \infty\mathsf{Grpd}). \tag{4.9}$$

We obtain a six-functor formalism

$$\mathrm{QCoh} : \mathrm{Corr}(\mathrm{Psh}((\mathsf{dC}^\infty\mathsf{Alg})^{\mathsf{op}}), E)^{\otimes} \to \mathsf{Pr}_{\mathsf{st}}^{L,\otimes} \tag{4.10}$$

such that the class $E \supseteq \mathrm{rep}$ is stable under disjoint unions, $*$-local on the target, $!$-local on the source, is tame and satisfies $E \subseteq \delta E$.

Example 4.1.9 (Rigid-analytic geometry)**.** *Let $K/\mathbf{Q}_p$ be a complete field extension considered as a non-Archimedean field, together with the category CBorn_K of non-Archimedean complete bornological K-vector spaces. We consider the $\mathbf{Q}^{>0}$-sorted Lawvere theory $\mathscr{T} = \mathsf{Tate}$ in which $\mathrm{Map}(\underline{\gamma}, \underline{\gamma}') := \mathrm{Hom}(K\langle \underline{x}'/\underline{\gamma}'\rangle, K\langle \underline{x}/\underline{\gamma}\rangle)$ are the morphisms of affinoid algebras (here $\underline{\gamma}, \underline{\gamma}'$ are tuples of elements of $\mathbf{Q}^{>0}$). There is an obvious functor*

$$\mathsf{Tate}^{\mathsf{op}} \to \mathsf{CAlg}(D_{\geqslant 0}(\mathsf{CBorn}_K)) \subseteq \mathsf{CAlg}(D(\mathsf{CBorn}_K)). \tag{4.11}$$

which is coproduct-preserving (by the flatness of Tate algebras with respect to $\widehat{\otimes}_K$). Therefore we may apply the above formalism with $\mathscr{V} = D(\mathsf{CBorn}_K)$ and we obtain a six-functor formalism

$$\mathrm{QCoh} : \mathrm{Corr}(\mathrm{Psh}(\mathsf{Alg}^{\mathsf{op}}_{\mathsf{Tate}}), E) \to \mathsf{Pr}^{L,\otimes}_{\mathsf{st}} \tag{4.12}$$

such that the class $E \supseteq \mathrm{rep}$ is stable under disjoint unions, $*$-local on the target, $!$-local on the source, is tame and satisfies $E \subseteq \delta E$.

Example 4.1.10. *More generally Example 4.1.9 still works if we replace the non-Archimedean field K with any non-Archimedean Banach ring[1] R. In particular we could take $R = (\mathbf{Z}, |\cdot|_{\mathrm{triv}})$ and we obtain a "six-functor formalism in universal non-Archimedean geometry". If we take $R = (\mathbf{Z}_p, |\cdot|_p)$ we obtain a "six-functor formalism for p-adic formal stacks". There are many more examples.*

Example 4.1.11 (Entire functional calculus)**.** *Consider the (Archimedean) Banach field $(\mathbf{C}, |\cdot|_\infty)$ and the category $\mathsf{CBorn}_{\mathbf{C}}$ of (Archimedean) complete bornological $\mathbf{C}$-vector spaces. We consider the Lawvere theory $\mathscr{T} = \mathsf{EFC}_{\mathbf{C}}$ in which*

$$\mathrm{Map}(m, n) := \mathrm{Hom}(\mathscr{O}^{\mathrm{hol}}(\mathbf{C}^n), \mathscr{O}^{\mathrm{hol}}(\mathbf{C}^m)). \tag{4.13}$$

Here Hom denotes the continuous morphisms of Fréchet $\mathbf{C}$-algebras, or equivalently bounded morphisms for the von Neumann bornology. There is an obvious functor

$$\mathsf{EFC}^{\mathsf{op}}_{\mathbf{C}} \to \mathsf{CAlg}(D_{\geqslant 0}(\mathsf{CBorn}_{\mathbf{C}})) \subseteq \mathsf{CAlg}(D(\mathsf{CBorn}_{\mathbf{C}})). \tag{4.14}$$

which is coproduct preserving (by the flatness of $\mathscr{O}^{\mathrm{hol}}(\mathbf{C})$ with respect to $\widehat{\otimes}_{\mathbf{C}}$). Therefore we may apply the above formalism with $\mathscr{V} = D(\mathsf{CBorn}_{\mathbf{C}})$. The category of algebras for this Lawvere theory is $\mathsf{EFCAlg}_{\mathbf{C}}$ and the ∞-category of derived algebras is the ∞-category of derived EFC-algebras:

$$\mathsf{dEFCAlg}_{\mathbf{C}} := \mathrm{Fun}^{\Pi}(\mathsf{EFC}^{\mathsf{op}}_{\mathbf{C}}, \infty\mathsf{Grpd}). \tag{4.15}$$

We obtain a six-functor formalism

$$\mathrm{QCoh} : \mathrm{Corr}(\mathrm{Psh}((\mathsf{dEFCAlg}_{\mathbf{C}})^{\mathsf{op}}), E)^{\otimes} \to \mathsf{Pr}^{L,\otimes}_{\mathsf{st}} \tag{4.16}$$

such that the class $E \supseteq \mathrm{rep}$ is stable under disjoint unions, $*$-local on the target, $!$-local on the source, is tame and satisfies $E \subseteq \delta E$.

4.2 Six-functor formalism for $\mathcal{D}^\infty$-modules associated to a Fermat theory

Now, inspired by [BK18] and [Tar25], we consider "six-functor formalisms for $\mathcal{D}^\infty$-modules". If $\mathscr{T}$ is a Lawvere theory we let $S \mapsto \mathrm{Free}_{\mathscr{T}}\{S\}$ denote the left adjoint to the forgetful functor $\mathsf{Alg}_{\mathscr{T}} \to \mathsf{Set}$. If Λ is a set and $\mathscr{T}$ is a Λ-sorted Lawevere theory we let $[\lambda : \Lambda \to \mathsf{Set}] \mapsto \mathrm{Free}_{\mathscr{T}}\{S/\underline{\lambda}\}$ denote the left adjoint to the forgetful functor $\mathsf{Alg}_{\mathscr{T}} \to \mathsf{Set}^{\Lambda}$.

[1]We recall that for a general Banach ring R we define $\mathsf{CBorn}_R := \mathsf{Ind}^m\mathsf{Ban}_R$, c.f. Remark 2.1.33.

Definition 4.2.1. *A* Fermat theory *is a Lawvere theory over* $\mathsf{Com}_{\mathbf{Z}}$ *satifying* Hadamard's Lemma. *Let* $\underline{z} := (z_1, \ldots, z_n)$*. Then for each* $f(x, \underline{z}) \in \mathrm{Free}_{\mathscr{T}}\{x, \underline{z}\}$ *we require that there exists a unique* $g \in \mathrm{Free}_{\mathscr{T}}\{x, y, \underline{z}\}$ *(called the* difference quotient*) such that*

$$f(x, \underline{z}) - f(y, \underline{z}) = (x - y)g(x, y, \underline{z}). \tag{4.17}$$

Remark 4.2.2. *When* Γ *is a partially ordered abelian group (meaning that* Γ *is an abelian group equipped with a translation-invariant partial order), one has a notion of* Γ*-sorted Fermat theory [BBKK24, Definition 4.2.1].*

The following result due to Dubuc and Kock [DK84] is fundamental.

Proposition 4.2.3. *Let* $\mathscr{T}$ *be a Fermat theory and let* $A \in \mathsf{Alg}_{\mathscr{T}}$*. Given any ring-theoretic ideal* I *in* A*, then* A/I *acquires the canonical structure of a* $\mathscr{T}$*-algebra such that* $A \to A/I$ *is a morphism of* $\mathscr{T}$*-algebras.*

Now using Proposition 4.2.3 we can define the notion of localization in an arbitrary $\mathscr{T}$-algebra.

Definition 4.2.4. *Let* $\mathscr{T}$ *be a Fermat theory and let* $A \in \mathsf{Alg}_{\mathscr{T}}$ *and* $a \in A$*. The* localization of A at a *is the quotient* $A\{a^{-1}\} := (A \coprod \mathrm{Free}_{\mathscr{T}}\{x\})/\langle xa - 1\rangle$*.*

Remark 4.2.5. *When* $\mathscr{T}$ *is a* Γ*-sorted Fermat theory (c.f. Remark 4.2.2) one can similarly define for each* $A \in \mathsf{Alg}_{\mathscr{T}}$*,* $a \in A$ *and* $\gamma \in \Gamma$ *the "rational localization"* $A\{a/\gamma\} := (A \coprod \mathrm{Free}_{\mathscr{T}}\{x/\gamma\})/\langle xa - 1\rangle$*.*

One can similarly define the localization at any collection of elements and this enjoys the expected universal property [CR13, §3.3.4].

Definition 4.2.6. *[BK18, Tar25] Let* $\mathscr{T}$ *be a Fermat theory.*

(i) Let $A \in \mathsf{Alg}_{\mathscr{T}}$ *with a ring-theoretic ideal* I*. We define the* $\mathscr{T}$-radical of I *to be*

$$\sqrt[\mathscr{T}]{I} := \langle a \in A : (A/I)\{a\} \cong \{0\}\rangle\,. \tag{4.18}$$

(ii) Let $A \in \mathsf{Alg}_{\mathscr{T}}$*. The* $\mathscr{T}$-nilradical of A *is defined to be* $\sqrt[\mathscr{T}]{0}$ *and the* $\mathscr{T}$*-reduction is defined to be* $A^{\mathscr{T}\mathrm{red}} := A/\sqrt[\mathscr{T}]{0} \in \mathsf{Alg}_{\mathscr{T}}$*.*

(iii) Let $X \in \mathrm{Psh}(\mathsf{dAlg}_{\mathscr{T}}^{\mathsf{op}})$*. We define the* $\mathscr{T}$-de Rham space of X *by* $X_{\mathscr{T}dR}(A) := X((\pi_0 A)^{\mathscr{T}\mathrm{red}})$*. We let* $(-)_{\mathscr{T}\mathrm{dR}}$ *denote the corresponding endofunctor of* $\mathrm{Psh}(\mathsf{dAlg}_{\mathscr{T}}^{\mathsf{op}})$*.*

We may drop the $\mathscr{T}$ *from all this notation when it is clear from the context.*

Remark 4.2.7. *When* $\mathscr{T}$ *is a* Γ*-sorted Fermat theory (c.f. Remarks 4.2.2 and 4.2.5) we change the definition of* $\mathscr{T}$*-radical to*

$$\sqrt[\mathscr{T}]{I} := \langle a \in A : \forall \gamma \in \Gamma, (A/I)\{a/\gamma\} \cong \{0\}\rangle\,. \tag{4.19}$$

Now let us continue with notations as in Theorem 4.1.6, but we assume in addition that $\mathscr{T}$ is a Fermat theory. The endofunctor $(-)_{\mathscr{T}\mathrm{dR}}$ of $\mathrm{Psh}(\mathsf{dAlg}_{\mathscr{T}}^{\mathsf{op}})$ evidently preserves all limits (and colimits), and so we obtain a symmetric-monoidal functor

$$(-)_{\mathscr{T}\mathrm{dR}} : \mathrm{Corr}(\mathrm{Psh}(\mathsf{dAlg}_{\mathscr{T}}^{\mathsf{op}}), E_{\mathscr{T}dR})^{\otimes} \to \mathrm{Corr}(\mathrm{Psh}(\mathsf{dAlg}_{\mathscr{T}}^{\mathsf{op}}), E)^{\otimes}, \tag{4.20}$$

here $E_{\mathscr{T}\mathrm{dR}}$ is the preimage of E under $(-)_{\mathscr{T}\mathrm{dR}}$. Hence by post-composition, we obtain a six-functor formalism

$$\mathrm{Crys}_{\mathscr{T}} := \mathrm{QCoh} \circ (-)_{\mathscr{T}\mathrm{dR}} : \mathrm{Corr}(\mathrm{Psh}(\mathsf{dAlg}_{\mathscr{T}}^{\mathsf{op}}), E_{\mathscr{T}dR})^{\otimes} \to \mathsf{Pr}_{\mathsf{st}}^{L,\otimes}. \tag{4.21}$$

This is the "six-functor formalism for $\mathscr{T}$-analytic $\mathcal{D}$-modules on $\mathscr{T}$-stacks".

Remark 4.2.8. *The above construction has an obvious generalization to Γ-sorted Lawvere theories, c.f. Remark 4.2.7.*

Example 4.2.9. *With notations as in Example 4.1.8. The Lawvere theory* CartSm *is a Fermat theory. The notion of reduction introduced above is the ∞-reduction functor* $R_\infty : \mathsf{C}^\infty\mathsf{Alg} \to \mathsf{C}^\infty\mathsf{Alg}$ *of Borisov–Kremnizer [BK18]. We obtain a six-functor formalism*

$$\mathrm{Crys}_{\mathsf{CartSm}} := \mathrm{QCoh} \circ (-)_{\mathrm{dR}} : \mathrm{Corr}(\mathrm{Psh}((\mathsf{dC}^\infty\mathsf{Alg})^{\mathsf{op}}), E_{\mathrm{dR}})^{\otimes} \to \mathsf{Pr}_{\mathsf{st}}^{L,\otimes}. \tag{4.22}$$

This is conjecturally a "six-functor formalism for $\mathcal{D}^\infty$-modules on derived C^∞-stacks". It is possible that these two Examples together with the results of §2.3.3 can inform the construction of chiral algebras.

Example 4.2.10. *With notations as in Example 4.1.9. The Lawvere theory* Tate *is a* $\mathbf{Q}^{>0}$*-sorted Fermat theory. Following Remarks 4.2.5 and 4.2.7 and the above construction we obtain a six-functor formalism*

$$\mathrm{Crys}_{\mathsf{Tate}} := \mathrm{QCoh} \circ (-)_{\mathrm{dR}} : \mathrm{Corr}(\mathrm{Psh}(\mathsf{Alg}_{\mathsf{Tate}}^{\mathsf{op}}), E_{\mathrm{dR}})^{\otimes} \to \mathsf{Pr}_{\mathsf{st}}^{L,\otimes}. \tag{4.23}$$

This is conjecturally a "six-functor formalism for $\widehat{\mathcal{D}}$-modules on derived Tate stacks".

Example 4.2.11. *With notations as in Example 4.1.11. The Lawvere theory* $\mathsf{EFC}_{\mathbf{C}}$ *is a Fermat theory. We obtain a six-functor formalism*

$$\mathrm{Crys}_{\mathsf{EFC}} := \mathrm{QCoh} \circ (-)_{\mathrm{dR}} : \mathrm{Corr}(\mathrm{Psh}((\mathsf{dEFCAlg}_{\mathbf{C}})^{\mathsf{op}}), E_{\mathrm{dR}})^{\otimes} \to \mathsf{Pr}_{\mathsf{st}}^{L,\otimes}. \tag{4.24}$$

This is conjecturally a "six-functor formalism for $\mathcal{D}^\infty$-modules in complex Stein geometry".

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