A six-functor formalism for quasi-coherent sheaves and crystals on rigid-analytic varieties

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What I will cover

- Much of the content of this talk is from a preprint I uploaded: "A six-functor formalism for quasi-coherent sheaves and crystals on rigid-analytic varieties" arXiv:2409.07592 [math.NT].

Plan of the talk

1. I will explain what a six-functor formalism is.
2. I will explain how to obtain a six-functor formalism in derived rigid-analytic geometry, using the approach of Ben-Bassat–Kelly–Kremnitzer.
3. I will explain how to use this to obtain a six-functor formalism for "analytic crystals" which is related to the $\widehat{\mathcal{D}}$-modules of Ardakov and Wadsley.

Philosophical discussion

- In analytic geometry, the notion of a quasi-coherent sheaf or crystal is necessarily a derived notion, because of the failure of flatness of affinoid localizations w.r.t. $\widehat{\otimes}$. (The naïve notion fails to satisfy descent, c.f. Gabber's example).
- For this reason, we systematically use higher category theory. In fact, this is also an advantage because "derived geometry" and "six-functor formalisms" are natively ∞-categorical notions.
- By using higher category theory one also gains access to the powerful machinery of presentable and stable categories, the Barr–Beck–Lurie theorem, adjoint functor theorems, Mathew's theory of descendable algebras...

What is a six-functor formalism? I

- We start with a category $\mathcal{C}$ of "geometric objects" X (admitting all fiber products). For instance we could have $\mathcal{C} = \text{Schemes}$, $\mathcal{C} = \text{LCHaus}$, $\mathcal{C} = \text{dRig}$.
- A six-functor formalism, roughly speaking, associates to each $X \in \mathcal{C}$ a closed symmetric monoidal ∞-category $(\mathcal{Q}(X), \otimes)$, in a manner which satisfies a very large number of functorial properties.

What is a six-functor formalism? II

We usually also single out a collection E of "special" or "!-able" edges in $\mathcal{C}$. The pair $(\mathcal{C}, E)$ is called a *geometric setup*.

- To each morphism $f\colon X \to Y$ of $\mathcal{C}$ we associate a symmetric monoidal "pullback" functor $f^* : \mathcal{Q}(Y) \to \mathcal{Q}(X)$.
- To each morphism $f\colon X \to Y$ in E we associate a "compactly supported pushforwards" $f_! : \mathcal{Q}(X) \to \mathcal{Q}(Y)$.
- For composable f, g we should have compatible isomorphisms $f_! g_! \simeq (fg)_!$ and $g^* f^* \simeq (fg)^*$.

This assigment should satisfy:

- *base-change*: $g^* f_! \simeq f'_! g'^{,*}$.
- *projection formula*: $f_! \otimes_Y \mathrm{id} \simeq f_!(\mathrm{id} \otimes_X f^*)$.
- The functors $(f^*, f_!, \otimes_X)$ admit right adjoints $(f_*, f^!, \underline{\mathrm{Hom}}_X)$, respectively.

Since the base change and projection formulas are themselves required to be compatible with the composition isomorphisms this leads to a potentially enormous number of things to check!

What is a six functor formalism? III

Remarkably, one can provide a succinct definition of a six-functor formalism via the *category of correspondences*.
The ∞-category $\mathrm{Corr}(\mathcal{C}, E)$ has:

- objects the same as those of $\mathcal{C}$
- morphisms $X \dashrightarrow Y$ given by spans $X \xleftarrow{g} U \xrightarrow{f} Y$ with $f \in E$. The composite of $X \leftarrow U \rightarrow Y$ and $Y \leftarrow V \rightarrow Z$ is given by the composed span $X \leftarrow U \leftarrow U \times_Y V \rightarrow V \rightarrow Z$.
- monoidal structure built from the coCartesian monoidal structure on $\mathcal{C}^{\mathrm{op}}$.

A lax-symmetric monoidal functor $\mathcal{Q} : \mathrm{Corr}(\mathcal{C}, E) \rightarrow \mathrm{Cat}_\infty$ determines functors

$$g^* := \mathcal{Q}(X \xleftarrow{g} Y = Y) : \mathcal{Q}(X) \rightarrow \mathcal{Q}(Y) \text{ and}$$
$$f_! := \mathcal{Q}(X = X \xrightarrow{f} Y) : \mathcal{Q}(X) \rightarrow \mathcal{Q}(Y) \text{ and}$$
$$\otimes_X : \mathcal{Q}(X) \times \mathcal{Q}(X) \rightarrow \mathcal{Q}(X).$$

What is a six-functor formalism? IV

Definition (Liu-Zheng, Gaitsgory-Rozenblyum, Mann)

A six-functor formalism on $(\mathcal{C}, E)$ is a lax-symmetric monoidal functor

$$\mathcal{Q} : \mathrm{Corr}(\mathcal{C}, E) \to \mathrm{Cat}_\infty$$

such that all the g^*, $f_!$, $\otimes_X$ admit right adjoints.

- This definition provides a viable way to manipulate six-functor formalisms and produce new ones out of old ones.
- Can streamline proof of complicated theorems e.g. Poincaré/Grothendieck-Verdier duality (Zavyalov '23).
- Six-functor formalisms can help to inform us what the "correct" definitions of some objects/functors should be, see (for instance) work of Heyer–Mann on smooth representations.

On the construction of six-functor formalisms

- One can often use [Man22, Prop A.5.12] to construct a basic six-functor formalism $\mathcal{Q}$ on $(\mathcal{C}, E_0)$ for some class E_0.
- Then following [Sch22, Thm 4.20] we developed an *extension formalism* for abstract six-functor formalisms, to extend $\mathcal{Q}$ to $(\mathcal{C}, E)$ for some larger class E with *good stability properties*:
  - being *local on the source*;
  - *local on the target*;
  - *stable under disjoint unions*;
  - and *tame*.
- The required extension is obtained as a transfinite composition of Mann's extension principles.
- Intuitively, this *extension formalism* is a systematic way to construct *compact supports* in such a way that the compactly supported pushforwards has the best possible categorical properties.
- Subsequently, a similar *extension formalism*, plus much more, has been developed in the recent work of Heyer–Mann.

Derived rigid geometry I

- Rigid analytic geometry was introduced by Tate ('71). In some ways it behaves similarly to complex-analytic geometry, but lives over p-adic fields $K/\mathbf{Q}_p$ rather than $\mathbf{C}$.
- In order to obtain a six-functor formalism for quasi-coherent sheaves, one has to "derive" rigid geometry, in order to obtain base-change.
- Our theory of derived rigid spaces follows Ben-Bassat–Kelly–Kremnitzer. In the spirit of Toën–Vezzosi and Deligne, (derived) analytic geometry is viewed as a precise analogue of (derived) algebraic geometry done *relative* to the symmetric monoidal ∞-category $D_{\geq 0}(\mathrm{CBorn}_K)$.
- The category CBorn_K of *complete bornological spaces* can be thought of as a substitute for locally convex K-vector spaces, but with better homological and algebraic properties. See Jack Kelly, "*Homotopy in exact categories*", Mem. Amer. Math. Soc. ('24).

Derived rigid geometry II

- We define a full category $\mathsf{dAfndAlg} \subseteq \mathsf{CAlg}(D_{\geq 0}(\mathsf{CBorn}_K))$ and $\mathsf{Afnd} := \mathsf{dAfndAlg}^{\mathsf{op}}$ whose objects are denoted $\mathsf{dSp}(A)$. We define the *weak Grothendieck topology*. The prestack

$$\mathsf{QCoh}(\mathsf{dSp}(A)) := \mathsf{Mod}_A(D(\mathsf{CBorn}_K))$$

satisfies descent in the weak topology.

- We define a certain full subcategory of *derived rigid spaces*:

$$\mathsf{dRig} \subseteq \mathsf{Shv}_{\mathrm{weak}}(\mathsf{dAfnd}, \infty\mathsf{Grpd}),$$

equipped with a *strong Grothendieck topology*. By Kan extension, QCoh extends to a sheaf on dRig.

- For $X \in \mathsf{dRig}$ the small site X_{strong} is really a locale, which is spatial, i.e., comes from a sober topological space $|X|$. The space $|X|$ is invariant under classical truncation. When $X = \mathsf{dSp}(A)$ we have

$$|X| \simeq |\,\mathsf{Spa}(\pi_0 A, (\pi_0 A)^\circ)|,$$

where the latter is the Huber spectrum.

Six-functor formalism for rigid spaces

Example

By proving base-change and the projection formula for qcqs morphisms "by hand" and then using [Man22, Prop A.5.12] we can extend QCoh to a basic six-functor formalism

$$\mathrm{QCoh} : \mathrm{Corr}(\mathrm{dRig}, qcqs) \to \mathrm{Pr}^{L}_{st}.$$

in which every qcqs morphism f satisfies $f_! = f_*$. By the *extension formalism* we can extend this to a six-functor formalism on (dRig, E) for a much larger class E.

- *A priori* it is not clear that E contains any interesting morphisms besides the qcqs ones.
- We show that a certain class of infinite covers is of *universal !-descent*.
- This can be used to show that every relative Stein space belongs to E.

Local cohomology.

- In order to place an analytic strucure on more general subsets $S \subseteq |X|$, we define the categories Pairs and Germs.

Definition

1. The category Pairs consists of objects (X, S), where $S \subseteq |X|$ is a closed subset such that $j : U \hookrightarrow X$ satisfies $j^! \xrightarrow{\sim} j^*$. (Here U is the complement of S.)
2. We define

$$\mathrm{QCoh}(X, S) := \Gamma_S \, \mathrm{QCoh}(X) \subseteq \mathrm{QCoh}(X),$$

as the full subcategory on objects supported along $S \subseteq |X|$.

- The inclusion admits both a right and a left adjoint:

$$\text{“local homology”} \quad \mathrm{L}_S \dashv \mathrm{incl}_S \dashv \Gamma_S \quad \text{“local cohomology”}.$$

- The category Germs is obtained by localizing Pairs with respect to an “obvious” system of morphisms.
- We will write $(X, S) \mapsto [(X, S)]$ for the image of (X, S) under the localization functor.

Crystals I

- Using the operations L_S and Γ_S one can obtain a six-functor formalism on Pairs and Germs.
- For example: for a morphism $f\colon (X,S) \to (Y,T)$ the upper-star functor is $\mathrm{L}_S f^*$ and the upper-shriek functor is $\Gamma_S f^!$.
- We take

$$\mathrm{PStk} := \mathrm{PSh}(\mathrm{qcqsGerms}, \infty\mathrm{Grpd})$$

  to be our ambient category of analytic prestacks. By Kan extension, one obtains a six-functor formalism QCoh on $(\mathrm{PStk}, \widetilde{E})$ for some class $\widetilde{E}$.
- For separated $f\colon X \to Y$, we define the "analytic" infinitesimal groupoid

$$\mathrm{Inf}(X/Y)_\bullet := [(X^{\bullet+1/Y}, |\Delta_{\bullet+1/Y}X|)] \in s\mathrm{PStk},$$

  and the relative "analytic" de Rham space

$$(X/Y)_{dR} := \varinjlim_{[n]\in\Delta^{op}} \mathrm{Inf}(X/Y)_n \in \mathrm{PStk}.$$

Crystals II

- We identify a class of *good* morphisms. The functor $(-)_{dR}$ preserves pullbacks of edges in good and $(-)_{dR}$ takes good into $\widetilde{E}$. Hence, it induces

$$(-)_{dR} : \mathsf{Corr}(\mathsf{qcsdRig}, \mathrm{good}) \to \mathsf{Corr}(\mathsf{PStk}, \widetilde{E}).$$

- By post-composition we obtain a six-functor formalism

$$\mathsf{Crys} := \mathsf{QCoh} \circ (-)_{dR}$$

  on $(\mathsf{qcsdRig}, \mathrm{good})$ which, by the extension formalism, can be extentded to a six-functor formalism on (dRig, E_{dR}) for some large class E_{dR}.
- The class good contains all open immersions and projections off smooth classical affinoids with free tangent bundle.

Crystals III

- Crys, viewed as a prestack on dRig, satisfies descent in the analytic topology.

Theorem (Kashiwara's equivalence)

If $i : Z \to X$ is a Zariski-closed immersion which is locally a neighbourhood retract, the the pair $(i_{dR,*}, i^*_{dR})$ induces an equivalence

$$\mathrm{Crys}(Z) \simeq \Gamma_Z \mathrm{Crys}(X),$$

where the latter is the full subcategory of objects supported along $Z \subseteq |X|$.

Monadicity

By definition, we have $\mathrm{Crys}(X) = \mathrm{QCoh}(X_{dR})$. We would like to understand this category better. There is a canonical morphism

$$p : X \to X_{dR}$$

which in fact satisfies $p_! \xrightarrow{\sim} p_*$. So we get an adjoint triple $p^* \dashv p_* \dashv p^!$:

$$\mathrm{QCoh}(X) \begin{array}{c} \xleftarrow{\quad p^! \quad} \\ \xrightarrow{\quad p_* \quad} \\ \xleftarrow{\quad p^* \quad} \end{array} \mathrm{QCoh}(X_{dR}).$$

Theorem (S.)

- *The adjunction $p^* \dashv p_*$ is comonadic.*
- *If X is a smooth affinoid with free tangent bundle then the adjunction $p_* \dashv p^!$ is monadic.*

So we can describe $\mathrm{QCoh}(X_{dR})$ as a category of comodules over the comonad $p^* p_*$ or modules over the monad $p^! p_*$.

Differential monad and jet comonad

Now we would like to understand the comonad $p^* p_*$ and the monad $p^! p_*$. We have a Cartesian square

$$\begin{array}{ccc} [(X \times X, \Delta X)] & \xrightarrow{\pi_1} & X \\ {\scriptstyle \pi_2}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ X & \xrightarrow[p]{} & X_{dR} \end{array}$$

and hence, by base-change, we obtain equivalences

$$p^! p_* \simeq \pi_{1,*} \Gamma_\Delta \pi_2^! \quad \text{and} \quad p^* p_* \simeq \pi_{2,*} \, \mathsf{L}_\Delta \, \pi_1^*.$$

Definition

- $\mathcal{D}^\infty_{X/K} := p^! p_* \simeq \pi_{1,*} \Gamma_\Delta \pi_2^!$ is called the *monad of differential operators*.
- $\mathcal{J}^\infty_{X/K} := p^* p_* \simeq \pi_{2,*} \, \mathsf{L}_\Delta \, \pi_1^*$. is called the *comonad of jets*.

A connection to work of Ardakov–Wadsley

Theorem

When X is a classical smooth affinoid with free tangent bundle, $\mathcal{D}^{\infty}_{X/K} 1_X \simeq \widehat{\mathcal{D}}_{X/K}(X)$ in $\mathrm{QCoh}(X)$, where the latter is the infinite-order differential operators of Ardakov-Wadsley (viewed as an object concentrated in degree 0).

Formulas for the six operations of Crys(X)

Theorem (S.)

Let $f : X \to Y$ be a morphism in dRig *(possibly* !*-able).*

(I) f^*_{dR} *is given by* $f^* : \mathrm{Comod}_{\mathcal{J}^\infty_{Y/K}} \to \mathrm{Comod}_{\mathcal{J}^\infty_{X/K}}$.

(II) $f_{dR,*}$ *is given by*

$$\varprojlim_{[n]\in\Delta} \mathcal{D}^\infty_{Y/K} f_*(\mathcal{J}^\infty_{X/K})^n : \mathrm{Comod}_{\mathcal{J}^\infty_{X/K}} \to \mathrm{Mod}_{\mathcal{D}^\infty_{Y/K}}.$$

(III) $f_{dR,!}$ *is given by*

$$\varinjlim_{[n]\in\Delta^{\mathrm{op}}} \mathcal{J}^\infty_{Y/K} f_!(\mathcal{D}^\infty_{X/K})^n : \mathrm{Mod}_{\mathcal{D}^\infty_{X/K}} \to \mathrm{Comod}_{\mathcal{J}^\infty_{Y/K}}.$$

(IV) $f^!_{dR}$ *is given by* $f^! : \mathrm{Mod}_{\mathcal{D}^\infty_{Y/K}} \to \mathrm{Mod}_{\mathcal{D}^\infty_{X/K}}$.

(V) *The tensor product on* $\mathrm{Comod}_{\mathcal{J}^\infty_{X/K}}$ *is given by that of* QCoh(X).

(VI) *We can also give a formula for the internal Hom (omitted).*

Future work

- By using the recent work of Heyer–Mann (and their construction of the category of kernels) and the theory of six-functor formalisms it should be possible to develop a good theory of Fourier–Mukai transforms in analytic geometry.
- We expect that there is a coalgebra structure (with respect to convolution) on $\mathrm{L}_\Delta 1_{X\times X}$ which induces the comonad $\mathcal{J}_X^\infty$.
- By choosing a system U_n of neighbourhoods of the diagonal ΔX in $X \times X$, one should be able to write $X_{dR} = \varinjlim_n X_{dR,n}$ in such a way that, for smooth X, the fibers of $p_n : X \to X_{dR,n}$ look like the open unit disk. Then p_n should be cohomologically smooth, so that the theory of FM transforms applies to $\mathcal{D}_X^n := p_n^! p_{n,!}$ and one obtains an equivalence

$$\mathrm{QCoh}(X_{dR}) \simeq \varprojlim_n \mathrm{Mod}_{\mathcal{D}_X^n 1_X} \mathrm{QCoh}(X).$$

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