$\mathrm{Gal}(\overline{E}/E)$ as a geometric fundamental group

Arun Soor

Abstract

These are notes for a learning seminar talk on diamonds. These notes basically lifted from other places. The main references are [Wei17] and [SW20]. All typos are my own!

1 Plan

Let $\mathbf{Q}_p \subseteq E \subseteq \mathbf{C}_p$ be complete field extensions with $E/\mathbf{Q}_p$ finite. Let $\pi \in o_E$ be a uniformizer, let $\mathbf{F}_q/\mathbf{F}_p$ be the residue field of E.

- Using Lubin-Tate theory we will construct the perfectoid open unit disk $\widetilde{\mathbf{D}}_{\mathbf{C}_p}$ over $\mathbf{C}_p$ which is an E-vector space object in perfectoid spaces. Then $\widetilde{\mathbf{D}}^*_{\mathbf{C}_p} := \widetilde{\mathbf{D}}_{\mathbf{C}_p} \setminus \{0\}$ carries an action of $E^\times$
- We will prove that the diamond $Z := \widetilde{\mathbf{D}}^{*,\diamond}_{\mathbf{C}_p}/E^\times$ has $\pi_1^{\text{ét}}(Z) \cong \mathrm{Gal}(\overline{E}/E) =: G_E$. Therefore G_E is a *geometric* fundamental group. Here the adjective *geometric* refers to the fact that $\widetilde{\mathbf{D}}^*_{\mathbf{C}_p}$ is defined over an algebraically closed field.

The proof roughly goes as follows.

- In addition to the obvious structural morphism to $\mathrm{Spa}(\mathbf{C}_p^\flat, o_{\mathbf{C}_p^\flat})$ the space $(\widetilde{\mathbf{D}}^*_{\mathbf{C}_p})^\flat$ also lives over the perfectoid field $E_\infty^\flat = \mathbf{F}_q(\!(X^{1/q^\infty})\!)$.
- One notices that the "unquotiented" adic Fargues–Fontaine curve is an untilt of $(\widetilde{\mathbf{D}}^*_{\mathbf{C}_p})^\flat$ over $E_\infty^\flat$:
$$(\widetilde{\mathbf{D}}^*_{\mathbf{C}_p})^\flat \cong (Y_E \widehat{\otimes} E_\infty)^\flat. \tag{1}$$
This is $o_E^\times$-equivariant ($o_E^\times$ acts on E_∞ by the Lubin–Tate character). The action of π on the right corresponds to $\varphi^{-1} \otimes 1$ on the left up to absolute Frobenius. This shows that
$$\widetilde{\mathbf{D}}^{*,\diamond}_{\mathbf{C}_p}/E^\times = (X_E)^\diamond \tag{2}$$
where $X_E = Y_E/\varphi^{\mathbf{Z}}$ is the Fargues–Fontaine curve.
- Therefore it suffices to classify finite étale covers of X_E. For this we will use the classification of vector bundles on the Fargues–Fontaine curve X_E together with the correspondence
$$\{\text{fét covers } f : X' \to X\} \leftrightarrow \{\text{fin. loc. free } \mathcal{O}_X\text{-algebras w. perfect trace pairing}\}.$$
sending $[f : X' \to X] \mapsto f_*\mathcal{O}_{X'}$ and inverse given by the relative spectrum. In other words we want to show that when $X = X_E$, every object on the right-hand has trivial underlying vector bundle.

As a corollary of the"Main Theorem" there is a correspondence

$$\{\text{connected fét deg } n \text{ covers of } \widetilde{\mathbf{D}}^{*,\diamond}_{\mathbf{C}_p}/E^\times\} \leftrightarrow \{\text{deg } n \text{ field extensions } E'/E\}. \tag{3}$$

Time permitting, I'll try to describe the cover on the left corresponding to an E'/E on the right.

2 Perfectoid spaces arising as adic generic fibers.

Lemma 2.1. *Let K be a nonarchimedean field with pseudo-uniformizer ϖ, let R be a flat o_K-algebra which is adic and complete for a f.g. ideal $I \ni \varpi$. Say $I = (f_1, \dots, f_r, \varpi)$. Set*

$$S_n := R\langle f_1^n/\varpi, \dots, f_r^n/\varpi\rangle, \quad R_n := S_n[1/\varpi], \quad R_n^+ := \text{int. clos.}(S_n \subseteq R_n). \tag{4}$$

Then $\mathrm{Spa}(R,R)_\eta = \varinjlim \mathrm{Spa}(R_n, R_n^+)$.

Proof. We will show that the functor of points is the same. Looking at the definition of $\mathrm{Spa}(R,R)_\eta$ as the fiber over $\mathrm{Spa}(K,o_K) \to \mathrm{Spa}(o_K,o_K)$, we see its (T,T^+) points are continuous o_K-linear homomorphisms $g : R \to T^+$. The $g(f_i)$ are topologically nilpotent and hence $g(f_i)^n \subseteq \varpi T^+$ for some n and all i. So g extends to $S_n \to T^+$ and we get $(R_n, R_n^+) \to (T, T^+)$ by taking completions and integral closures and inverting ϖ (in the right order). □

Example 2.2. $\mathrm{Spa}(o_K[\![X]\!], o_K[\![X]\!])_\eta$ *is the rigid open unit disk.*

Lemma 2.3. *Let K be a perfectoid field of characteristic 0 with pseudo-uniformizer ϖ. Let R be an o_K-algebra which is adic and complete for a f.g. ideal I. Assume that R/ϖ is semiperfect. Then $(\mathrm{Spa}(R,R))^\flat_{\eta^\flat}$ and $\mathrm{Spa}(R,R)_\eta$ are perfectoid and*

$$\mathrm{Spa}(R^\flat, R^\flat)_{\eta^\flat} = (\mathrm{Spa}(R,R))^\flat_{\eta^\flat} = (\mathrm{Spa}(R,R)_\eta)^\flat. \tag{5}$$

Proof. I will give the covers by affinoid perfectoids without justfication. For details see [Wei17]. Let $f_1, \dots, f_r$ be generators for an ideal of definition of $R^\flat$. The cover is given as in Lemma 2.1:

$$\mathrm{Spa}(R^\flat, R^\flat)_{\eta^\flat} = \varinjlim \mathrm{Spa}(R_n^\flat, R_n^{\flat,+}), \tag{6}$$

where $R_n^\flat = R^\flat\langle f_i^n/\varpi\rangle[1/\varpi]$, and

$$\mathrm{Spa}(R,R)_\eta = \varinjlim \mathrm{Spa}(R_n, R_n^+), \tag{7}$$

where $R_n = R\langle f_i^{\sharp,n}/\varpi\rangle[1/\varpi]$. Further, one has $\mathrm{Spa}(R_n, R_n^+)^\flat = \mathrm{Spa}(R_n^\flat, R_n^{\flat,+})$. □

3 The perfectoid open unit disk

Lubin–Tate theory. Let $\phi(X) \in o_E[\![X]\!]$, be a Frobenius power series (meaning $\phi = \pi X (\bmod X)$ and $\phi = X^q (\bmod \pi)$). Let $F_\phi(X,Y) \in o_E[\![X,Y]\!]$ be the corresponding Lubin–Tate formal group law with $\phi \in \mathrm{End}(F_\phi)$. Then there is $[.]_\phi : o_E \to \mathrm{End}(F_\phi)$ with $[\pi]_\phi = \phi$ and $[a]_\phi = aX + \dots$. We define

$$\mathcal{F}_n := \{z \in \mathfrak{m}_{\mathbf{C}_p} : [\pi^n]_\phi(z) = 0\}, \quad E_n := E(\mathcal{F}_n), \tag{8}$$

and we define E_∞ to be the completion of $\bigcup_n E_n$. Lubin–Tate theory asserts that the Tate module $\mathcal{F}_\infty := \varprojlim_n \mathcal{F}_n$ is free of rank 1 as an o_E-module so that the choice of basis element determines a character $\chi_E : \mathrm{Gal}(E_\infty/E) \to o_E^\times$, which turns out to be an isomorphism. The field E_∞ is perfectoid with tilt

$$E_\infty^\flat = \mathbf{F}_q(\!(X^{1/q^\infty})\!), \text{ and } o_{E_\infty^\flat} = \mathbf{F}_q[\![X^{1/q^\infty}]\!]. \tag{9}$$

Example 3.1. *When $E = \mathbf{Q}_p$ one takes $\pi = p$, $\phi = (1+X)^p - 1$, then $F_\phi(X,Y) = (X+1)(Y+1) - 1$ is the multplicative law and $[a]_\phi = (1+X)^a - 1$ for $a \in \mathbf{Z}_p$. We obtain $\mathcal{F}_n = \{\zeta - 1 : \zeta^{p^n} = 1\}$ and $E_n = \mathbf{Q}_p(\zeta_{p^n})$, and χ_E is the cyclotomic character. (One can do almost the same thing for unramified extensions $E/\mathbf{Q}_p$).*

The perfectoid open unit disk. We define $\mathbf{D}_E := \mathrm{Spa}(o_E[\![X]\!], o_E[\![X]\!])_\eta$ and

$$R_E := \big(\varinjlim_{\phi} o_E[\![X]\!]\big)^\wedge_{(\pi,X)} \quad \text{so that} \quad \mathrm{Spa}(R_E, R_E) = \varprojlim_{\mathrm{Spa}(\phi)} \mathrm{Spa}(o_E[\![X]\!], o_E[\![X]\!]). \tag{10}$$

Then $\widetilde{\mathbf{D}}_E := \mathrm{Spa}(R_E, R_E)_\eta$ is an E-vector space objects in adic spaces. If $\widetilde{\mathbf{D}}_{\mathbf{C}_p}$ denotes the base-change of $\widetilde{\mathbf{D}}_E$ from E to $\mathbf{C}_p$ then $\widetilde{\mathbf{D}}_{\mathbf{C}_p}$ then by Lemma 2.3 one has

- $\widetilde{\mathbf{D}}_{\mathbf{C}_p} = \mathrm{Spa}(R_{\mathbf{C}_p}, R_{\mathbf{C}_p})_\eta$ is perfectoid and $(\widetilde{\mathbf{D}}_{\mathbf{C}_p})^\flat = \mathrm{Spa}(R^\flat_{\mathbf{C}_p}, R^\flat_{\mathbf{C}_p})_{\eta^\flat}$.

In order to make the connection to the Fargues–Fontaine curve later, let us describe this tilt a little more explicitly. The special fiber $\mathrm{Spa}(R_E, R_E)_s$ equals

$$\varprojlim_{(.)^q} \mathrm{Spa}(\mathbf{F}_q[\![X]\!], \mathbf{F}_q[\![X]\!]) \cong \mathrm{Spa}(\mathbf{F}_q[\![X^{1/q^\infty}]\!], \mathbf{F}_q[\![X^{1/q^\infty}]\!]), \tag{11}$$

or if you prefer $\mathbf{F}_q \widehat{\otimes}_{o_E} R_E = \mathbf{F}_q[\![X^{1/q^\infty}]\!]$. We use this isomorphism in the second line below:

$$\begin{aligned} \mathrm{Spa}(o_{\mathbf{C}_p^\flat} \widehat{\otimes}_{\mathbf{F}_q} \mathbf{F}_q[\![X^{1/q^\infty}]\!]) &\cong \varprojlim_{(.)^q} \mathrm{Spa}(o_{\mathbf{C}_p^\flat}/\varpi o_{\mathbf{C}_p^\flat} \widehat{\otimes}_{\mathbf{F}_q} \mathbf{F}_q[\![X^{1/q^\infty}]\!]) \\ &\cong \varprojlim_{(.)^q} \mathrm{Spa}(R_{\mathbf{C}_p}/\varpi R_{\mathbf{C}_p}) = \mathrm{Spa}(R^\flat_{\mathbf{C}_p}). \end{aligned} \tag{12}$$

In the first line we used that $\mathbf{F}_q[\![X^{1/q^\infty}]\!]$ is perfect. In other words, passing to generic fibers and deleting 0, we obtained an isomorphism

$$\mathrm{Spa}(o_{\mathbf{C}_p^\flat} \widehat{\otimes}_{\mathbf{F}_q} o_{E^\flat_\infty})_\eta \setminus \{\pi = \varpi = 0\} \cong \widetilde{\mathbf{D}}^{*,\flat}_{\mathbf{C}_p}. \tag{13}$$

This isomorphism is equivariant for the action of $o_E^\times$ (on the left from tilting the Lubin–Tate character, and on the right from tilting the $o_E^\times$ action). There is a minor subtlety in matching up the Frobenius on $o_{\mathbf{C}_p^\flat}$ with the action of π on the right [Wei17, Lemma 4.0.9]. Conceptually, we've given a different presentation of the diamond $\widetilde{\mathbf{D}}^{*,\diamond}_{\mathbf{C}_p}/E^\times$.

4 The Fargues–Fontaine curve.

We recall the construction of the Fargues–Fontaine curve from Ken's talk. Actually, it can be constructed as the generic fiber of a formal scheme. We may define $B^{b,+} := W_{o_E}(o_{\mathbf{C}_p^\flat})$ where the subscript o_E denotes ramified Witt vectors, equipped with the usual $(\pi, [\varpi])$-adic topology. Then $B^{b,+}$ is an o_E-algebra and, as is usual for Witt vector constructions, is equipped with a Frobenius endomorphism coming from the Frobenius on $o_{\mathbf{C}_p^\flat}$. We define

$$Y_E := \mathrm{Spa}(B^{b,+}, B^{b,+})_\eta \setminus \{\pi = [\varpi] = 0\} \quad \text{and} \quad X_E := Y_E/\varphi^{\mathbf{Z}}. \tag{14}$$

We note that $(W_{o_E}(o_{\mathbf{C}_p^\flat}) \otimes_{o_E} o_{E_\infty})/\pi = o_{\mathbf{C}_p^\flat} \widehat{\otimes}_{\mathbf{F}_q} o_{E_\infty}/\pi$, so that by taking $\varinjlim_{(.)^q}$ we obtain $(B^{b,+} \widehat{\otimes}_{o_E} o_{E_\infty})^\flat = o_{\mathbf{C}_p^\flat} \otimes_{\mathbf{F}_q} o_{E^\flat_\infty}$. Hence by Lemma 2.3 and the previous section one has

$$(Y_E \widehat{\otimes}_E E_\infty)^\flat = \mathrm{Spa}(o_{\mathbf{C}_p^\flat} \widehat{\otimes}_{\mathbf{F}_q} o_{E^\flat_\infty})_\eta \setminus \{\pi = \varpi = 0\} \cong \widetilde{\mathbf{D}}^{*,\flat}_{\mathbf{C}_p}, \tag{15}$$

via an $o_E^\times$-equivariant isomorphism. We will continue to ignore the minor subtlety in matching up the φ and π actions. Hence we obtain an isomorphism

$$\pi_1^{\text{ét}}(\widetilde{\mathbf{D}}^{*,\diamond}_{\mathbf{C}_p}/E^\times) \cong \pi_1^{\text{ét}}(X_E). \tag{16}$$

4.1 The Fargues–Fontaine curve is geometrically simply connected

Finally, we will prove that $\pi_1^{\text{ét}}(X_E) \cong G_E$. For this we recall the Dieudonné–Manin classification of vector bundles on X_E. We let $\breve{E} := W_{o_E}(\overline{\mathbf{F}}_q)[1/\pi]$ denote the maximal unramified extension of E. Note that Y_E lies over $\mathrm{Spa}(\breve{E})$ and the Frobenius endomorphism φ of Y_E lies over that of $\breve{E}$, so we get $X_E \to \breve{E}/\varphi^{\mathbf{Z}}$. Therefore, we can pull back vector bundles on $\breve{E}/\varphi^{\mathbf{Z}}$.

Vector bundles on $\breve{E}/\varphi^{\mathbf{Z}}$ are called *isocrystals*. This is (by definition) the same as a finite-dimensional $\breve{E}$-vector space M equipped with the data of an isomorphism $\varphi_M : \varphi^* M \xrightarrow{\sim} M$. The pullback of (M, φ_M) amounts to the φ-equivariant vector bundle $(\mathcal{O} \otimes_{\breve{E}} M, \varphi \otimes \varphi_M)$ on Y_E which we regard as a vector bundle $\mathcal{E}(M)$ on X_E.

The category of isocrystals is semisimple and there is a bijection $\lambda \mapsto M(\lambda)$ between (reduced) rationals and isomorphism classes of simples. For such we define $\mathcal{O}(\lambda) := \mathcal{E}(M(-\lambda))$. For example $\mathcal{O}(0) = \mathcal{O}_{X_E}$. Now we can state the classifcation of vector bundles on the curve:

Theorem 4.1.
- *Every vector bundle $\mathcal{E}$ on X_E is isomorphic to a vector bundle of the form $\bigoplus_{i=1}^n \mathcal{O}(\lambda_i)$ for a unique sequence of rational numbers $\lambda_1 \le \cdots \le \lambda_n$.*
- *If $\lambda_i = d_i/r_i$ in reduced form then*

$$\mathcal{O}(\lambda_1) \otimes \mathcal{O}(\lambda_2) \cong \mathcal{O}(\lambda_1 + \lambda_2)^{\oplus(r_1, r_2)}. \tag{17}$$

- *The global sections are*

$$H^0(X_E, \mathcal{O}(\lambda)) \cong \begin{cases} \text{a Banach–Colmez space} & \text{if } \lambda > 0, \\ E, & \text{if } \lambda = 0, \\ 0, & \text{if } \lambda < 0. \end{cases} \tag{18}$$

Theorem 4.2. *The Fargues–Fontaine curve is geometrically simply connected.*

Proof. Let $X := X_E$ and $\mathcal{O} := \mathcal{O}_X$. As observed at the beginning of the talk, it suffices to show that every finite locally free $\mathcal{O}$-algebra $\mathcal{A}$ with perfect trace pairing $\mathcal{A} \otimes \mathcal{A} \to \mathcal{O}$ has trivial underlying vector bundle. Let us say that $\mathcal{A} = \bigoplus_i \mathcal{O}(\lambda_i)$ for $\lambda_1 \le \cdots \le \lambda_s$. Because the vector bundle $\mathcal{A}$ is self–dual one has $\deg(\mathcal{A}) = \deg(\mathcal{A}^\vee) = -\deg(\mathcal{A})$, which reads $(\prod_i r_i)(\sum_i \lambda_i) = -(\prod_i r_i)(\sum_i \lambda_i)$ and so $\sum_i \lambda_i = 0$. Let $\lambda := \lambda_s \ge 0$. Assume (for a contradiction) that $\lambda > 0$. The multiplication gives $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ which restricts to $\mathcal{O}(\lambda) \otimes \mathcal{O}(\lambda) \to \mathcal{A}$ which gives a global section

$$f \in \mathrm{Hom}(\mathcal{O}(\lambda) \otimes \mathcal{O}(\lambda), \mathcal{A}) = H^0(X, \mathcal{A} \otimes \mathcal{O}(-2\lambda))^{\oplus r}, \tag{19}$$

which must be zero because the latter has all negative slopes. We conclude that every $f \in H^0(X, \mathcal{O}(\lambda)) \subseteq H^0(X, \mathcal{A})$ satisfies $f^2 = 0$. Because $\mathcal{A}$ is a finite étale $\mathcal{O}$-algebra, $H^0(X, \mathcal{A})$ is reduced we must then have $H^0(X, \mathcal{O}(\lambda)) = 0$, so $\lambda < 0$, contradiction.

So all $\lambda_i = 0$ and the underlying vector bundle of $\mathcal{A}$ is trivial. $\square$

5 Explicit description of finite étale covers of $\widetilde{\mathbf{D}}^{*,\diamond}_{\mathbf{C}_p}/E^\times$

The previous constructions depended on the field E. We will no longer suppress this dependence in our notation, so let $Z_E := \widetilde{\mathbf{D}}^{*,\diamond}_{E,\mathbf{C}_p}/E^\times$.

We will show that a finite extension E'/E of fields induces a "norm" morphism of diamonds $N_{E'/E} : Z_{E'} \to Z_E$ which fits into a commutative square

$$\begin{tikzcd} \pi_1^{\text{ét}}(Z_{E'}) \arrow[r, hook] \arrow[d] & \pi_1^{\text{ét}}(Z_{E'}) \arrow[d] \\ G_{E'} \arrow[r, hook] & G_E \end{tikzcd} \tag{20}$$

The following sketch can be found in the Berkeley notes. Let $n = [E' : E]$. We note that $\mathbf{D}_{\breve{E}'}$ is a rigid p-divisible group of height n and dimension 1. By the work of [Hed10] we may take its "nth exterior power" which is a rigid p-divisible group of dimension 1 and height one, so isomorphic to $\widehat{\mathbf{G}}_{m,o_{\breve{E}'}}$ and so we obtain an alternating map $\lambda : \mathbf{D}^n_{\breve{E}'} \to \widehat{\mathbf{G}}_{m,o_{\breve{E}'}}$. If one chooses an o_E-basis $\alpha_1, \dots, \alpha_n$ for $o_{E'}$ then we may define $N_{E'/E}(x) := \lambda(\alpha_1 x, \dots, \alpha_n x)$ which gives $N_{E'/E} : \mathbf{D}_{E',\mathbf{C}_p} \to \mathbf{D}_{E,\mathbf{C}_p}$ after base change. By construction $N_{E'/E}(\alpha x) := N_{E'/E}(\alpha) N_{E'/E}(x)$ so we get $N_{E'/E} : \widetilde{\mathbf{D}}_{E',\mathbf{C}_p} \to \widetilde{\mathbf{D}}_{E,\mathbf{C}_p}$ which is equivariant for the norm map $E' \to E$, so finally we get $Z_{E'} \to Z_E$.

References

[Hed10] Seyed Mohammad Hadi Hedayatzadeh. *Exterior Powers of Barsotti-Tate Groups.* PhD thesis, ETH Zurich, 2010.

[SW20] Peter Scholze and Jared Weinstein. *Berkeley lectures on p-adic geometry*, volume 207 of *Ann. Math. Stud.* Princeton, NJ: Princeton University Press, 2020.

[Wei17] Jared Weinstein. Gal$(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ as a geometric fundamental group. *Int. Math. Res. Not. IMRN*, (10):2964–2997, 2017.