Period morphism and tower

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Abstract

These are notes for a learning seminar talk on moduli spaces of p-divisible groups given in February 2023.

Contents

1 RZ data of type EL

We introduce a slightly different perspective for the moduli problem of type (E). Main reference for this section is [RV14]. Let $F/\mathbb{Q}_p$ be finite, set $K_0 = \breve{F}$ with Frobenius $\sigma \in \mathrm{Gal}(K_0/F)$, let B be a finite dimensional semisimple F-algebra with maximal order O_B, let V be a finite dimensional B-module. In this case the *associated algebraic group* is $G = \mathrm{GL}_B(V)$ viewed as a functor on $\mathbb{Q}_p$-algebras by

$$G(R) = \mathrm{GL}_{B\otimes_{\mathbb{Q}_p} R}(V \otimes_{\mathbb{Q}_p} R). \tag{1}$$

We let $b \in G(K_0)$ and let $\mu : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$ be a cocharacter. Let $\Lambda \subseteq V$ be an O_B-stable lattice. This induces an integral model $\mathcal{G} = \mathrm{GL}_{O_B}(\Lambda)$ of G over $\mathbb{Z}_p$. We only consider b up to σ-conjugacy, i.e., $b \sim b'$ iff $b' = g^{-1} b\sigma(g)$ for some $g \in G(K_0)$, write $[b]$ for its σ-conjugacy class. We view μ as defined up to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-conjugacy and write $\{\mu\}$ for the class and $E = E_{\{\mu\}}$ for its field of definition. Then $E/\mathbb{Q}_p$ is finite and called the *Shimura field*. Associated to $b \in B$ we also define the functor J_b:

$$J_b(R) = \{g \in G(R \otimes_{\mathbb{Q}_p} K_0) : g(b\sigma) = (b\sigma)g\}, \tag{2}$$

it only depends on $[b]$ up to isomorphism. Then

$$(F, B, O_B, V, [b], \{\mu\}, \Lambda) \tag{3}$$

is a simple integral RZ datum of type EL, if the following conditions are satisfied:

(i) The pair (b, μ) is admissible. This means that the filtered isocrystal $(V \otimes K_0, b \otimes \sigma, V_K^i)$ is admissible[1], i.e., belongs to the essential image of Fontaine's functor, i.e., there is a crystalline Galois representation U over $\mathbb{Q}_p$ with $V \otimes K_0 \cong (U \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\mathrm{G}_K}$ as filtered isocrystals, c.f. [RZ96, 1.6].

[1] Here $K = K_0E$ is finite over K_0 and the filtration on $V_K = V \otimes K$ is by the weights of μ.

(ii) The isocrystal $(V \otimes K_0, b\sigma)$ has slopes in $[0,1]$.

(iii) The weights of μ on V_K are only 0 and 1.

Note that $J_b(\mathbb{Q}_p)$ is the B-linear automorphisms of $(V \otimes K_0, b\sigma)$. We explain how this is related to the moduli problem of type (E) previously introduced. We take $F = \mathbb{Q}_p$. For $S \in \mathrm{Nilp}_{O_{\breve{E}}}$, we consider pairs (X, ι) where $\iota : O_B \to \mathrm{End}(X)$ and the Kottwitz condition[2]:

$$\mathrm{char}(\iota(\beta); \mathrm{Lie}(X)) = \mathrm{char}(\beta; V_0), \quad \text{for all } \beta \in O_B, \tag{4}$$

where $V_0 \subseteq V \otimes \overline{\mathbb{Q}}_p$ is the weight 0 part for any $\mu \in \{\mu\}$; its isoclass as a B-module is defined over E.

The condition (ii) says that $(V \otimes K_0, b\sigma)$ is isomorphic to the isocrystal of some p-divisible group over $\overline{\mathbb{F}}_p$. Fix $b \in [b]$. We choose a $\mathbb{X}$ over $\overline{\mathbb{F}}_p$ such that its isocrystal[3] $E(\mathbb{X})$ is B-equivariantly isomorphic to $(V \otimes K_0, b\sigma)$. $\mathbb{X}$ is called the framing object. Then the functor

$$\begin{aligned} \breve{\mathcal{M}} : \mathrm{Nilp}_{O_{\breve{E}}} &\to \mathrm{Sets} \\ S &\mapsto \{\text{triples } (X, \iota, \varrho)\}/\cong, \end{aligned} \tag{5}$$

where $\varrho : \mathbb{X} \times_{\overline{\mathbb{F}}_p} \overline{S} \to X \times_S \overline{S}$ is an O_B-equivariant quasi-isogeny, is representable by a formal scheme $\breve{\mathcal{M}}$ locally formally of finite type over $\mathrm{Spf}(O_{\breve{E}})$. From the definitions, we have $J_b(\mathbb{Q}_p) = \mathrm{Aut}(E(\mathbb{X})) = \mathrm{Isog}(D(\mathbb{X}), D(\mathbb{X})) = \mathrm{Qisog}(\mathbb{X}, \mathbb{X})$. It follows that $g \in J_b(\mathbb{Q}_p)$ acts on $\breve{\mathcal{M}}$ by

$$g : (X, \iota, \varrho) \mapsto (X, \iota, \varrho \circ g^{-1}). \tag{6}$$

A technical remark about when $\breve{\mathcal{M}}$ is nonempty. All the conditions (i)-(iii) are satisfied if there is a finite extension K/K_0 and a p-divisible group $\widetilde{\mathbb{X}}$ over the ring of integers O_K of K, whose reduction $\widetilde{\mathbb{X}}_{\overline{\mathbb{F}}_p}$ is equipped with a quasi-isogeny $\mathbb{X} \to \widetilde{\mathbb{X}}_{\overline{\mathbb{F}}_p}$ such that:

- The weight filtration from μ, and the Hodge filtration on $M^{\mathrm{rig}}(\widetilde{\mathbb{X}}) :=$ (the global sections of $M^{\mathrm{rig}}_{\widetilde{\mathbb{X}}}$ over $\mathrm{Spm}(O_K)$) coincide. Taking $\mathcal{M} = \mathrm{Spf}(O_K)$ in Proposition 2.1, the given quasi-isogeny induces an isomorphism $E(\mathbb{X}) \otimes K = V \otimes K \cong M^{\mathrm{rig}}(\widetilde{\mathbb{X}})$. Then the Hodge filtration

$$0 \to \mathrm{Fil}^1 \to M^{\mathrm{rig}}(\widetilde{\mathbb{X}}) \to \mathrm{Lie}(\widetilde{\mathbb{X}}) \to 0 \tag{7}$$

and the weight filtration induced by the cocharacter μ

$$0 \to V_1 \to V \otimes K \to V/V_1 \to 0 \tag{8}$$

should agree.

[2] This is understood as an equality of polynomial functions in $\mathcal{O}_S$ - since O_B preserves Λ, the right side here is a polynomial with coefficients in O_E and we compare it to the left side via the structure morphism, see [RV14, §4.2].

[3] We always work with covariant objects, i.e., $E(\mathbb{X})$ is the Lie algebra of the universal vector extension of $\mathbb{X}$.

2 Period morphism

Reference for this section is [Wan09, §2.4] or [RZ96, Ch. 5]. Let S_0 be a scheme with p locally nilpotent. Recall that we defined the covariant Dieudonné functor

$$\mathbb{D} : \mathrm{BT}_{S_0} \to \left\{\text{Crystals in } \mathcal{O}_{S_{0,\mathrm{Cris}}} - \text{modules locally free of finite rank}\right\}. \tag{9}$$

If $S_0 \hookrightarrow S$ is a nilpotent pd thickening then by Grothendieck-Messing theory, the functor

$$\begin{aligned} \mathrm{GM} : \mathrm{BT}_S &\to \{\text{admissible pairs } (G_0, \mathrm{Fil}^1 \hookrightarrow \mathbb{D}(G_0)_{(S_0 \hookrightarrow S)})\} \\ G &\mapsto (G_0 = G|_{S_0}, V(G) \hookrightarrow \mathbb{D}(G_0)_{(S_0 \hookrightarrow S)}) \end{aligned} \tag{10}$$

is an equivalence of categories and compatible with base change in S. Recall that a pair is called admissible if Fil^1 is a locally direct summand which reduces to $V(G_0) \hookrightarrow \mathrm{Lie}(E(G_0))$ on S_0. Now (F, O, k, π) will be a complete DVR of characteristic $(0, p)$ and k perfect. Let $\mathcal{M}$ be a formal scheme locally formally of finite type over $\mathrm{Spf}(O)$. Such $\mathcal{M}$ are π-adically complete. Let $\mathcal{M}_0'$ be the closed subscheme defined by $p\mathcal{O}_{\mathcal{M}}$. Then (9) can be used to define a functor

$$\begin{aligned} \mathbb{D} : \mathrm{BT}_{\mathcal{M}} &\to \{\text{“admissible” extensions of locally free finite rank } \mathcal{O}_{\mathcal{M}}\text{-modules}\} \\ X &\mapsto (0 \to \mathrm{Fil}^1 \to M_X \to \mathrm{Lie}(X) \to 0). \end{aligned} \tag{11}$$

moreover the formation of M_X is compatible with base change in X. Let $\mathcal{M}_n'$ be the subscheme defined by $p^{n+1}\mathcal{O}_{\mathcal{M}}$. Then $\mathcal{M}_0' \hookrightarrow \mathcal{M}_n'$ is a nilpotent pd-thickening, with respect to the canonical divided power structure $\gamma_m(p) = p^m/m!$ on $p^{n+1}\mathcal{O}_{\mathcal{M}}$. That is to say, $\mathcal{M}_0' \hookrightarrow \mathcal{M}_n'$ is an object of $\mathrm{Cris}(\mathcal{M}_0')$, and therefore by Grothendieck-Messing theory we obtain admissible filtrations

$$0 \to V(X_n) \to \mathbb{D}(X_0)_{(\mathcal{M}_0' \hookrightarrow \mathcal{M}_n')} \to \mathrm{Lie}(X_n) \to 0, \tag{12}$$

here $X_n = X_{\mathcal{M}_n'}$. By compatibility of Grothendieck-Messing theory with base change we have $V(X_{n+1}) \otimes_{\mathcal{O}_{\mathcal{M}_{n+1}'}} \mathcal{O}_{\mathcal{M}_n'} = V(X_n)$. In particular Mittag-Leffler criterion is satisfied and we obtain the desired exact sequence

$$0 \to \varprojlim V(X_n) \to \varprojlim \mathbb{D}(X_0)_{(\mathcal{M}_0' \hookrightarrow \mathcal{M}_n')} \to \varprojlim \mathrm{Lie}(X_n) \to 0, \tag{13}$$

which is usually written as

$$0 \to \mathrm{Fil}^1 \to M_X \to \mathrm{Lie}(X) \to 0. \tag{14}$$

If $\mathbb{X}$ is a p-divisible group over $\overline{\mathbb{F}}_p$ then $M_{\mathbb{X}}$ is just the usual covariant Dieudonné module $D(\mathbb{X})$ over $W(\overline{\mathbb{F}}_p)$, in particular $M_{\mathbb{X}}[p^{-1}] = E(\mathbb{X})$ is the isocrystal associated to X. Now let $\mathcal{M}_0$ be the k-scheme defined by an ideal of definition of $\mathcal{M}$ containing π and let $\mathbb{X}$ be a fixed p-divisible group over k and with a fixed quasi-isogeny

$$\varrho : \mathbb{X}_{\mathcal{M}_0} \to X_{\mathcal{M}_0}. \tag{15}$$

Proposition 2.1. *[RZ96, Proposition 5.15] The quasi-isogeny ϱ induces a canonical and functorial isomorphism of locally free $\mathcal{O}_{\mathcal{M}^{\mathrm{rig}}}$-modules of finite rank, compatible with base change,*

$$\tilde{\varrho} : E(\mathbb{X}) \otimes_{W(k)[p^{-1}]} \mathcal{O}_{\mathcal{M}^{\mathrm{rig}}} \xrightarrow{\sim} M_X^{\mathrm{rig}}. \tag{16}$$

Proof. Due to some technicality with the rigid generic fiber functor we first treat the case where $\mathcal{M}$ is a topologically finitely presented (tfp) π-adic formal scheme. Let $\mathcal{M}_0, \mathcal{M}'_0$ be as above, then $\mathcal{M}_0 \subset \mathcal{M}'_0$ is a nilpotent immersion. By the rigidity of quasi-isogenies, ϱ induces a unique quasi-isogeny of p-divisible groups over $\mathcal{M}'_0$:

$$\varrho' : \mathbb{X}_{\mathcal{M}'_0} \to X_{\mathcal{M}'_0}. \tag{17}$$

Since the construction of M as in (13) only depends on the special fiber (functorially), the quasi-isogeny ϱ' induces an isogeny

$$M_{\mathbb{X}_{\mathcal{M}'_0}} \to M_X. \tag{18}$$

By the compatibility of Grothendieck-Messing theory with base change, this is identified with the isogeny $D(\mathbb{X}) \otimes_{W(k)} \mathcal{O}_{\mathcal{M}} \to M_X$ which becomes an isomorphism $E(\mathbb{X}) \otimes_{W(k)[p^{-1}]} \mathcal{O}_{\mathcal{M}^{\mathrm{rig}}} \xrightarrow{\sim} M_X^{\mathrm{rig}}$ after inverting p. For the general case of $\mathcal{M}$ locally formally of finite type, we may assume that $\mathcal{M} = \mathrm{Spf}(A)$ is affine. Then $\mathcal{M}^{\mathrm{rig}} = \varinjlim \mathrm{Spm}(B_n \otimes F)$ for some π-adic B_n with $\mathrm{Spf}(A) = \varinjlim \mathrm{Spf}(B_n)$. So the desired isomorphism $\tilde{\varrho} : E(\mathbb{X}) \otimes \mathcal{O}_{\mathcal{M}^{\mathrm{rig}}} \to M_X^{\mathrm{rig}}$ is obtained as a limit of the just constructed maps

$$E(\mathbb{X}) \otimes \mathcal{O}_{\mathrm{Spm}(B_n \otimes F)} \to M_{X_n}^{\mathrm{rig}}, \tag{19}$$

where X_n is the base change of X to $\mathrm{Spf}(B_n)$. □

We now return to the setting of §1. Let $(X^{\mathrm{univ}}, \iota^{\mathrm{univ}}, \varrho^{\mathrm{univ}})$ be the universal p-divisible group over $\breve{\mathcal{M}}$. Then by Proposition 2.1, the middle term in

$$0 \to (\mathrm{Fil}^1)^{\mathrm{rig}} \to M_X^{\mathrm{rig}} \to \mathrm{Lie}(X^{\mathrm{univ}}) \to 0 \tag{20}$$

of vector bundles on $\breve{\mathcal{M}}$ becomes (using $\tilde{\varrho}^{\mathrm{univ}}$):

$$0 \to (\mathrm{Fil}^1)^{\mathrm{rig}} \to V \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathcal{M}^{\mathrm{rig}}} \to \mathrm{Lie}(X^{\mathrm{univ}}) \to 0 \tag{21}$$

and hence determines a morphism $\breve{\pi} : \breve{\mathcal{M}} \to \mathrm{Gr}_{h-d}(V \otimes \breve{E})^{\mathrm{an}}$, the analytification of the Grassmannian parametrising $h-d$-dimensional planes in $V \otimes \breve{E}$. The period morphism $\breve{\pi}$ is étale, hence $\breve{M}^{\mathrm{rig}}$ is smooth over $\mathrm{Spm}(O_{\breve{E}})$ (c.f. [Wan09, Theorem 2.4.5] or [RZ96, Prop 5.17]).

3 Tower

We continue in this setting. Main reference for this section is [FM04, §2.3.9]. Recall that $\mathcal{G}$ is the integral model of G with $\mathcal{G}(\mathbb{Z}_p) = \mathrm{GL}_{O_B}(\Lambda)$. The Tate module $T_p(X^{\mathrm{univ}})$ of X^{univ} defines a $\mathbb{Z}_p$-local system of rank $h = \mathrm{ht}(\mathbb{X})$ on $(\breve{\mathcal{M}}^{\mathrm{rig}})_{\text{ét}}$ endowed with O_B-action, whose fiber $T_p(X_x^{\mathrm{univ}})$ over $x \in \breve{\mathcal{M}}^{\mathrm{rig}}(\overline{\breve{E}})$ is isomorphic to Λ. Let $Y \to \breve{\mathcal{M}}^{\mathrm{rig}}$ be a rigid analytic space. As a local system on Y with fiber Λ, we can associate to $T_p(X^{\mathrm{univ}})$ a cocycle c in $\check{Z}^1(Y, \mathcal{G}(\mathbb{Z}_p))$. A trivialisation modulo K of c is an element $\overline{\eta} \in \check{C}^0(Y, \mathcal{G}(\mathbb{Z}_p))/\check{C}^0(Y, K)$ such that $d\eta = c$ (modulo K). For such a trivialisation we may choose a particular representative η and write

$$\eta : \Lambda \xrightarrow{\sim} T_p(X^{\mathrm{univ}})\ [K], \tag{22}$$

square brackets $[K]$ emphasising the "modulo K". In this way the set of trivialisations of $T_p(X^{\mathrm{univ}})$ modulo K defines a sheaf of finite sets on $(\breve{\mathcal{M}}^{\mathrm{rig}})_{\text{ét}}$ which is equivalent data to a finite étale covering $\breve{\mathcal{M}}_K$ of $\breve{\mathcal{M}}^{\mathrm{rig}}$. This gives a tower $(\breve{\mathcal{M}}_K)_K$ of rigid analytic spaces over $\breve{\mathcal{M}}^{\mathrm{rig}}$ with $K \subset K'$ inducing $\breve{\mathcal{M}}_{K'} \to \breve{\mathcal{M}}_K$, which is Galois with group K/K' if K' is normal in K. If $g \in G(\mathbb{Q}_p)$ is such that $g^{-1}Kg \subseteq \mathcal{G}(\mathbb{Z}_p)$ then (we will show) g induces an isomorphism $\breve{\mathcal{M}}_K \xrightarrow{\sim} \breve{\mathcal{M}}_{g^{-1}Kg}$ naturally in g and K. Thus $G(\mathbb{Q}_p)$ acts on the tower. The action of $J_b(\mathbb{Q}_p)$ on the base extends to a "horizontal" action on the $\breve{\mathcal{M}}_K$, and we get a $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$-action on the tower.

We now describe isomorphism induced by g. We only do this on L-points for L/K_0 finite. A point $x \in \breve{\mathcal{M}}_K(L)$ corresponds to some $((X, \varrho), \overline{\eta})$, where X is over $\mathrm{Spf}(O_L)$. Since (we are assuming) $g^{-1}Kg \subset \mathcal{G}(\mathbb{Z}_p)$, the lattice $g\Lambda$ is stable under K and $\eta(g\Lambda)$ is stable for the action of $\mathrm{Gal}(\overline{L}/L)$ on $T_p(X^{\mathrm{rig}})$. It follows [Tat67, Proposition 12] that $\eta(g\Lambda)$ induces a p-divisible group X' with a quasi-isogeny $f : X \to X'$ such that under the composite

$$\varphi : \Lambda \otimes \mathbb{Q}_p \xrightarrow{\cong \eta \otimes 1} V_p(X) \xrightarrow{\cong f_*} V_p(X'), \tag{23}$$

we have $\varphi^{-1}(T_p(X')) = g\Lambda$. Then we obtain ϱ' by composing ϱ with the special fiber of f. Finally $\overline{\eta}' := g^{-1} \circ \varphi^{-1}$ gives a trivialisation modulo $g^{-1}Kg$ of $T_p(X')$, i.e., we have constructed

$$\begin{aligned} \breve{\mathcal{M}}_K(L) &\to \breve{\mathcal{M}}_{g^{-1}Kg}(L) \\ ((X, \varrho), \overline{\eta}) &\mapsto ((X', \varrho'), \overline{\eta}'). \end{aligned} \tag{24}$$

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