A six-functor formalism for analytic $\mathcal{D}$-modules on rigid-analytic varieties.

Arun Soor

Mathematical Institute, University of Oxford

Seminario de Geometría No Conmutativa del Atlántico Sur, 13 June 2024

What is a six-functor formalism? I

- We start with a category $\mathcal{C}$ of "geometric objects" X (admitting all fiber products). For instance we could have $\mathcal{C} = \text{Schemes}$, $\mathcal{C} = \text{LCHaus}$, $\mathcal{C} = \text{Rig}$.
- A six-functor formalism, roughly speaking, associates to each $X \in \mathcal{C}$ a closed symmetric monoidal ∞-category $(\mathcal{Q}(X), \otimes)$, in a manner which satisfies a very large number of functorial properties.
- This idea was initiated by Grothendieck in his study of functorial properties of ℓ-adic cohomology. It has recieved much attention recently as the precise definition of a six-functor formalism has been formulated and simplified, (c.f. work of Liu-Zheng, Gaitsgory-Rozenblyum, Mann, Scholze).

What is a six-functor formalism? II

We usually also single out a collection E of "special" or "!-able" edges in $\mathcal{C}$. The pair $(\mathcal{C}, E)$ is called a *geometric setup*.

- To each morphism $f\colon X \to Y$ of $\mathcal{C}$ we associate a symmetric monoidal "pullback" functor $f^* : \mathcal{Q}(Y) \to \mathcal{Q}(X)$.
- To each morphism $f\colon X \to Y$ in E we associate a "compactly supported pushforwards" $f_! : \mathcal{Q}(X) \to \mathcal{Q}(Y)$.
- For composable f, g we should have compatible isomorphisms $f_! g_! \simeq (fg)_!$ and $g^* f^* \simeq (fg)^*$.

This assigment should satisfy:

- *base-change*: $g^* f_! \xrightarrow{\sim} f'_! g'^{,*}$.
- *projection formula*: $f_! \otimes_Y \mathrm{id} \xrightarrow{\sim} f_!(\mathrm{id} \otimes_X f^*)$.
- The functors $(f^*, f_!, \otimes_X)$ admit right adjoints $(f_*, f^!, \underline{\mathrm{Hom}}_X)$, respectively.

Since the base change and projection formulas are themselves required to be compatible with the composition isomorphisms this leads to a potentially enormous number of things to check!

What is a six functor formalism? III

Remarkably, one can provide a succinct definition of a six-functor formalism via the *category of correspondences*.
The ∞-category $\mathrm{Corr}(\mathcal{C}, E)$ has:

- objects the same as those of $\mathcal{C}$
- morphisms $X \dashrightarrow Y$ given by spans $X \stackrel{g}{\leftarrow} U \stackrel{f}{\rightarrow} Y$ with $f \in E$. The composite of $X \leftarrow U \rightarrow Y$ and $Y \leftarrow V \rightarrow Z$ is given by the composed span $X \leftarrow U \leftarrow U \times_Y V \rightarrow V \rightarrow Z$.
- monoidal structure built from the Cartesian monoidal structure on $\mathcal{C}$.

A lax-symmetric monoidal functor $\mathcal{Q} : \mathrm{Corr}(\mathcal{C}, E) \rightarrow \mathrm{Cat}_\infty$ determines functors

$$g^* := \mathcal{Q}(X \stackrel{g}{\leftarrow} Y = Y) : \mathcal{Q}(X) \rightarrow \mathcal{Q}(Y) \text{ and}$$
$$f_! := \mathcal{Q}(X = X \stackrel{f}{\rightarrow} Y) : \mathcal{Q}(X) \rightarrow \mathcal{Q}(Y) \text{ and}$$
$$\otimes_X : \mathcal{Q}(X) \times \mathcal{Q}(X) \rightarrow \mathcal{Q}(X).$$

What is a six-functor formalism? IV

Definition (Liu-Zheng, Gaitsgory-Rozenblyum, Mann)

A six-functor formalism on $(\mathcal{C}, E)$ is a lax-symmetric monoidal functor

$$\mathcal{Q} : \mathrm{Corr}(\mathcal{C}, E) \to \mathrm{Cat}_\infty$$

such that all the g^*, $f_!$, $\otimes_X$ admit right adjoints.

- This definition provides a viable way to manipulate six-functor formalisms and produce new ones out of old ones.
- Can streamline proof of complicated theorems e.g. Poincaré/Grothendieck-Verdier duality (Zavyalov '23).
- Six-functor formalisms can help to inform us what the "correct" definitions of some objects/functors should be, e.g., one might understand ULA sheaves as "f-smooth objects".

Rigid analytic geometry

- Rigid analytic geometry was introduced by Tate ('71). In some ways it behaves similarly to complex-analytic geometry, but relative to p-adic fields $K/\mathbb{Q}_p$ rather than $\mathbb{C}$.
- Let $\mathrm{Rig}_{s,\mathrm{steady}}$ be the category of all separated rigid analytic spaces over K and *steady* morphisms (I will say what *steady* means later).
- The aim of this talk is to describe how to produce a six-functor formalism

$$\mathsf{Crys} : \mathsf{Corr}(\mathrm{Rig}_{s,\mathrm{steady}}, E) \to \mathsf{Cat}_\infty,$$

  where E is a collection of morphisms with good stability properties.
- For a smooth rigid-analytic space X/K the objects of the category $\mathsf{Crys}(X)$ are similar to modules over the sheaf $\widehat{\mathcal{D}}_{X/K}$ considered by Ardakov and Wadsley.
- This is similar in spirit to work of Rodríguez–Camargo ('24) and inspired by work of Andy Jiang ('23).

Quasicoherent sheaves I

1. Let $X = \mathrm{Sp}(A)$ be an affinoid rigid space corresponding to a K-affinoid algebra A. This is a K-Banach algebra. We view $A \in \mathrm{CBorn}_K$ as a monoid in the closed symmetric monoidal category of complete bornological K-vector spaces.
2. We let $\mathrm{Mod}(A)$ be the quasi-abelian category of modules over the monoid. We rely crucially on work of Jack Kelly ('21) to obtain a model structure on the unbounded chain complexes $Ch(\mathrm{Mod}(A))$ such that the underlying ∞-category

$$\mathrm{QCoh}(\mathrm{Sp}(A)) := N(Ch(\mathrm{Mod}(A)))[W^{-1}]$$

   is stable, presentable and (closed) symmetric monoidal.
3. For each morphism $\mathrm{Sp}(B) \to \mathrm{Sp}(A)$ we consider the derived pullback $B\widehat{\otimes}^{\mathbf{L}}_A -$ and obtain a prestack

$$\mathrm{QCoh} : \mathrm{Afnd}^{\mathrm{op}} \to \mathrm{Comm}(\mathrm{Pr}^L_{\mathrm{st}})$$

   where the latter is the category of presentably symmetric monoidal stable ∞-categories with left-adjoint functors.

Quasicoherent sheaves II

Theorem

- *The prestack* QCoh *is a sheaf in the weak G-topology on* Afnd*. Kan extension along* Afnd $\to$ Rig *makes* QCoh *into a sheaf on* Rig *equipped with the strong G-topology.*
- *For a morphism* $f: X \to Y$ *in* Rig*, the induced pullback functor* f^* *admits a right adjoint* f_**. If f is quasi-compact then* f_* *preserves colimits, commutes with restrictions to admissible opens, and satisfies the projection formula* $f_* \otimes_Y \mathrm{id} \xrightarrow{\sim} f_*(\mathrm{id} \otimes_X f^*)$*.*

A necessary condition for the assignment $X \to \mathrm{QCoh}(X)$ to extend to a six-functor formalism is to have base-change isomorphisms. It is well known that this is false in general; there are two solutions:

- Enhance Rig to some category of derived rigid spaces;
- or, restrict the class of morphisms to *steady* morphisms.

We will adopt the latter approach.

Quasicoherent sheaves III

The notion of a *steady morphism* is borrowed from Mann ('22).

Definition

A morphism $f\colon \mathrm{Sp}(B) \to \mathrm{Sp}(A)$ of affinoid rigid spaces is called *steady* if for all morphisms $g : \mathrm{Sp}(C) \to \mathrm{Sp}(A)$ the natural morphism $B\widehat{\otimes}^{\mathbf{L}}_A C \to B\widehat{\otimes}_A C$ is an isomorphism.
A morphism $f\colon X \to Y$ of rigid spaces is called *steady* if it is steady locally on the source and target.

- The inclusion $U \hookrightarrow X$ of an admissible open subset, is steady (Ben–Bassat-Kremnizer '17). The structure morphism $X \to \mathrm{Sp}\, K$ is always steady. Steady morphisms have good stability properties.
- Their importance is the following: if $g : X \to Y$ is steady then for any quasi-compact $f\colon Y' \to Y$ there is a base-change isomorphism

$$g^* f_* \xrightarrow{\sim} f'_* g'^{,*}.$$

Quasicoherent sheaves IV

With the definition of a steady morphism we can apply the results of Mann to obtain a six-functor formalism

$$\mathsf{QCoh} : \mathsf{Corr}(\mathsf{Rig}_{s,\mathrm{steady}}, qc) \to \mathsf{Cat}_\infty,$$

where qc is the class of quasi-compact morphisms. By a formal procedure taken from "Theorem 4.20" in Scholze's six-functor formalism notes, we can lift this to a six-functor formalism

$$\mathsf{QCoh} : \mathsf{Corr}(\mathsf{Rig}_{s,\mathrm{steady}}, E) \to \mathsf{Cat}_\infty,$$

where $E \supset qc$ is a much larger class of morphisms with good stability properties.

Local cohomology I

These ideas were inspired by Andy Jiang ('23). A classical theory of local cohomology was developed by Kisin ('99).

- Let $S \subset X$ be a subset such that the complement $U := X \setminus S$ is an admissible open. Let $j\colon U \to X$ be the inclusion. We impose the hypothesis that

$$j^! \xrightarrow{\sim} j^*.$$

- This gives rise to a category $\mathrm{Pairs}_{s,\mathrm{steady}}$. The objects are pairs (X, S) as above and a morphism $f\colon (X, S) \to (Y, T)$ is a morphism $f\colon X \to Y$ in $\mathrm{Rig}_{s,\mathrm{steady}}$ with $f(S) \subseteq T$.
- We define

$$\mathrm{QCoh}((X, S)) := \Gamma_S(\mathrm{QCoh}(X)) \subseteq \mathrm{QCoh}(X) \qquad (1)$$

as the full subcategory on objects M such that $j^* M \simeq 0$.

Local cohomology II

We make the important observation that the tautological inclusion

$$\mathrm{incl}_S : \Gamma_S(\mathrm{QCoh}(X)) \to \mathrm{QCoh}(X)$$

admits a right adjoint *and* a left adjoint:

$$\begin{aligned} \mathrm{incl}_S &\dashv \Gamma_S \qquad \text{“local cohomology”} \\ i_S^{-1} &\dashv \mathrm{incl}_S \qquad \text{“inverse image”}. \end{aligned}$$

With these additional operations we can lift QCoh to a six-functor formalism on $\mathrm{Pairs}_{qcs,\mathrm{steady}}$:

$$\mathrm{QCoh} : \mathrm{Corr}(\mathrm{Pairs}_{qcs,\mathrm{steady}}, all) \to \mathrm{Cat}_\infty\,.$$

For example: for a morphism $f : (X, S) \to (Y, T)$ the upper-star functor is $i_S^{-1} f^*$ and the upper-shriek functor is $\Gamma_S f^!$.

The category of germs

The category $\mathrm{QCoh}((X,S))$ does not depend on the whole ambient space X. We can formalise this notion using the category of *germs* (Berkovich '93).

Definition (Berkovich)

We define a system Φ of morphisms of $\mathrm{Pairs}_{s,\mathrm{steady}}$ as follows:

- A morphism $\varphi : (X,S) \to (Y,T)$ belongs to Φ if it induces an isomorphism of X with a neighbourhood of T in Y.
- The category $\mathrm{Germs}_{s,\mathrm{steady}}$ is defined to be the localization of $\mathrm{Pairs}_{s,\mathrm{steady}}$ at the class Φ.
- We will write $(X,S) \mapsto [(X,S)]$ for the image of (X,S) under the localization functor.

The six-functor formalism for $\mathrm{Pairs}_{qcs,\mathrm{steady}}$ then induces a six-functor formalism:

$$\mathrm{QCoh} : \mathrm{Corr}(\mathrm{Germs}_{qcs,\mathrm{steady}}, \mathit{all}) \to \mathrm{Cat}_\infty . \tag{2}$$

Stacks

- We can then take

$$\mathsf{Psh}(\mathsf{Germs}_{qcs,\mathrm{steady}}) := \mathsf{Psh}(\mathsf{Germs}_{qcs,\mathrm{steady}}, \infty - \mathsf{Grpd})$$

as our ∞-category of rigid analytic stacks. By Kan extension along the ∞-categorical Yoneda embedding, and "Theorem 4.20" of Scholze again, we can extend QCoh to a six-functor formalism

$$\mathsf{QCoh} : \mathsf{Corr}(\mathsf{Psh}(\mathsf{Germs}_{qcs,\mathrm{steady}}), \widetilde{E}) \to \mathsf{Cat}_{\infty},$$

where $\widetilde{E}$ is a collection of morphisms with good stability properties.
- By working with presheaves, we can now take arbitrary colimits of geometric objects. For instance, we can define quotient objects.

Crystals

- For $X \in \mathrm{Rig}_{s,\mathrm{steady}}$ and $n \geq 0$ we can consider the germ $[(X^{n+1}, \Delta X)]$ along the diagonal. These can be arranged into a simplicial object $[(X^{\bullet+1}, \Delta X)]$. Our analytic de Rham stack is defined to be:

$$X_{dR} := \varinjlim_{[n]\in\Delta^{\mathrm{op}}} [(X^{n+1}, \Delta X)]$$

  where the colimit is taken in $\mathrm{Psh}(\mathrm{Germs}_{qcs,\mathrm{steady}})$.
- The functor $X \mapsto X_{dR}$ is fiber-product preserving. Therefore it induces a functor

$$(-)_{dR} : \mathrm{Corr}(\mathrm{Rig}_{s,\mathrm{steady}}, E) \to \mathrm{Corr}(\mathrm{Psh}(\mathrm{Germs}_{qcs,\mathrm{steady}}), \widetilde{E}).$$

  By post-composition, we obtain a six-functor formalism

$$\mathrm{Crys} : \mathrm{Corr}(\mathrm{Rig}_{s,\mathrm{steady}}, E) \to \mathrm{Cat}_\infty.$$

Monadicity

By definition, we have $\mathrm{Crys}(X) = \mathrm{QCoh}(X_{dR})$. We would like to understand this category better. There is a canonical morphism

$$p : X \to X_{dR}$$

which in fact satisfies $p_! \xrightarrow{\sim} p_*$. So we get an adjoint triple $p^* \dashv p_* \dashv p^!$:

$$\mathrm{QCoh}(X) \begin{array}{c} \xleftarrow{\ p^!\ } \\ \xrightarrow{\ p_*\ } \\ \xleftarrow{\ p^*\ } \end{array} \mathrm{QCoh}(X_{dR}).$$

Theorem (S.)

- *The adjunction $p^* \dashv p_*$ is comonadic.*
- *The adjunction $p_* \dashv p^!$ is monadic.*

So we can describe $\mathrm{QCoh}(X_{dR})$ as a category of comodules over the comonad $p^* p_*$ or modules over the monad $p^! p_*$.

Differential monad and jet comonad

Now we would like to understand the comonad $p^* p_*$ and the monad $p^! p_*$. We have a Cartesian square

$$\begin{array}{ccc} [(X \times X, \Delta X)] & \xrightarrow{\pi_1} & X \\ \downarrow{\scriptstyle \pi_2} & \lrcorner & \downarrow{\scriptstyle p} \\ X & \xrightarrow[p]{} & X_{dR} \end{array}$$

and hence, by base-change, we obtain isomorphisms

$$p^! p_* \simeq \pi_{1,*} \Gamma_\Delta \pi_2^! \quad \text{and} \quad p^* p_* \simeq \pi_{2,*} i_\Delta^{-1} \pi_1^*.$$

Definition

- $\mathcal{D}_{X/K}^\infty := \pi_{1,*} \Gamma_\Delta \pi_2^!$ is called the *monad of differential operators*.
- $\mathcal{J}_{X/K}^\infty := \pi_{2,*} i_\Delta^{-1} \pi_1^*$ is called the *comonad of jets*.

A connection to work of Ardakov-Wadsley

Theorem (S.)

When X is a smooth affinoid with free tangent bundle, $\mathcal{D}^{\infty}_{X/K} 1_X \simeq \widehat{\mathcal{D}}_{X/K}(X)$ in $\mathrm{QCoh}(X)$, where the latter is the infinite-order differential operators of Ardakov-Wadsley (viewed as an object concentrated in degree 0).

Formulas for the six operations of Crys(X)

Theorem (S.)

Let $f\colon X \to Y$ *be a morphism in* $\mathrm{Rig}_{s,\mathrm{steady}}$.

(I) f^*_{dR} *is given by* $f^* : \mathrm{Comod}_{\mathcal{J}^\infty_{Y/K}} \to \mathrm{Comod}_{\mathcal{J}^\infty_{X/K}}$.

(II) $f_{dR,*}$ *is given by*

$$\varprojlim_{[n]\in\Delta} \mathcal{D}^\infty_{Y/K} f_*(\mathcal{J}^\infty_{X/K})^n : \mathrm{Comod}_{\mathcal{J}^\infty_{X/K}} \to \mathrm{Mod}_{\mathcal{D}^\infty_{Y/K}}.$$

(III) $f_{dR,!}$ *is given by*

$$\varinjlim_{[n]\in\Delta^{\mathrm{op}}} \mathcal{J}^\infty_{Y/K} f_!(\mathcal{D}^\infty_{X/K})^n : \mathrm{Mod}_{\mathcal{D}^\infty_{X/K}} \to \mathrm{Comod}_{\mathcal{J}^\infty_{Y/K}}.$$

(IV) $f^!_{dR}$ *is given by* $f^! : \mathrm{Mod}_{\mathcal{D}^\infty_{Y/K}} \to \mathrm{Mod}_{\mathcal{D}^\infty_{X/K}}$.

(V) *The tensor product on* $\mathrm{Comod}_{\mathcal{J}^\infty_{X/K}}$ *is given by that of* $\mathrm{QCoh}(X)$.

(VI) *We can also give a formula for the internal Hom (omitted).*

References I

Konstantin Ardakov and Simon Wadsley.
$\widehat{\mathcal{D}}$-modules on rigid analytic spaces I.
J. Reine Angew. Math., 747:221–275, 2019.

Oren Ben-Bassat and Kobi Kremnizer.
Non-archimedean analytic geometry as relative algebraic geometry.
Ann. Fac. Sci. Toulouse Math. (6), 26(1), 2017.

Vladimir G. Berkovich.
Étale cohomology for non-Archimedean analytic spaces.
Inst. Hautes Études Sci. Publ. Math., (78):5–161, 1993.

Juan Esteban Rodríguez Camargo.
Analytic de Rham stacks and derived rigid geometry, 2023.
Available at the author's webpage.

References II

Dennis Gaitsgory and Nick Rozenblyum.
A study in derived algebraic geometry. Vol. I. Correspondences and duality, volume 221 of *Mathematical Surveys and Monographs*.
American Mathematical Society, Providence, RI, 2017.

Andy Jiang.
The Derived Ring of Differential Operators, March 2023.
arXiv:2303.16083 [math].

Jack Kelly.
Homotopy in exact categories.
Mem. Amer. Math. Soc. (to appear), July 2021.
arXiv:1603.06557 [math].

Mark Kisin.
Analytic functions on Zariski open sets, and local cohomology.
J. Reine Angew. Math., 506:117–144, 1999.

References III

Yifeng Liu and Weizhe Zheng.
Enhanced six operations and base change theorem for higher Artin stacks, September 2017.
arXiv:1211.5948 [math].

Lucas Mann.
A p-adic 6-functor formalism in rigid-analytic geometry, June 2022.
arXiv:2206.02022 [math].

Peter Scholze.
Six-Functor Formalisms.
Available at `https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf`, 2022.

References IV

Arun Soor.
Quasicoherent sheaves for dagger analytic geometry, November 2023.
arXiv:2311.03101 [math].

Arun Soor.
Rigid analytic crystals and $\mathcal{D}$-modules.
In preparation, 2024.

Bogdan Zavyalov.
Poincaré Duality in abstract 6-functor formalisms, October 2023.
arXiv:2301.03821 [math].