Galois representation associated to a weight 2 modular form.

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1 Aim

The aim of the talk is to prove the following. For the most part we will be following the appendix by Brian Conrad from [RS01] very closely, which we also direct the reader to for all the techincal details.

Theorem 1.1. *Let ℓ be a prime, let $N > 5$ and let f be a weight 2 normalised eigenform, new at some level $M|N$. Let K_f be the number field of f and let $\mathfrak{l}$ be a place of K_f above ℓ. Then there exists a continuous representation $\varrho_{N,\mathfrak{l},f} : G_{\mathbb{Q}} \to GL_2(K_{f,\mathfrak{l}})$ unramified at all $p \nmid N\ell$ and such that the endomorphism $\varrho_{N,\mathfrak{l},f}(\mathrm{Frob}_p)$ satisfies the polynomial relation*

$$X^2 - a_p(f) + p\chi_f(p) = 0, \tag{1}$$

where $a_p(f)$ is the p^{th} Fourier coefficient and χ_f is the Nebentypus character.

2 Eichler-Shimura congruence relation

Let $Y_1(N)$ be the moduli functor on $\mathsf{Sch}_{/\mathbb{Z}[1/N]}$ given by[1]

$$Y_1(N)(S) := \left\{ \begin{array}{c} \text{isomorphism classes } (E, P) \text{ of elliptic curves } E/S \\ \text{with a section } P : S \to E[N] \text{ of exact order } N \end{array} \right\}. \tag{2}$$

If $N > 5$ then $Y_1(N)$ is representable by an affine scheme which is finite flat over the j-line $\mathbb{A}^1_{\mathbb{Z}[1/N]}$ (in fact finite étale over the locus where $j, j - 1728$ are both invertible). The compactified modular curve $X_1(N) \to \mathbb{P}^1_{\mathbb{Z}[1/N]}$ is defined as the normalization of $Y_1(N) \to \mathbb{A}^1_{\mathbb{Z}[1/N]}$. The curve $X_1(N)$ is proper over $\mathbb{Z}[1/N]$ and has good reduction at all primes $q \nmid N$. Similarly, we can define a moduli functor[2]

$$Y_1(N,p)(S) := \left\{ \begin{array}{c} \text{isomorphism classes } (E, P, C) \text{ of elliptic curves } E/S \\ \text{with a section } P : S \to E[N] \text{ of exact order } N \\ \text{and an order } p \text{ finite locally free sub-}S\text{-group scheme } C \subset E \\ \text{which does not meet the subgroup generated by } P \end{array} \right\}. \tag{3}$$

which is represented by an affine scheme over $\mathbb{Z}[1/Np]$, and has a compactification $X_1(N,p)$ which is proper over $\mathbb{Z}[1/Np]$. There is a correspondence of $\mathbb{Z}[1/Np]$-schemes:

$$\begin{gathered} X_1(N)_{/\mathbb{Z}[1/Np]} \xleftarrow{\pi_1^{(p)}} X_1(N,p) \xrightarrow{\pi_2^{(p)}} X_1(N)_{/\mathbb{Z}[1/Np]}, \\ (E,P) \hookleftarrow (E,P,C) \mapsto (E/C, P). \end{gathered} \tag{4}$$

[1] "Exact order N" means that it is a point of exact order N on geometric fibers.
[2] "Does not meet" means that it does not meet the subgroup generated by P, on geometric fibers.

in which the maps $\pi_1^{(p)}, \pi_2^{(p)}$ are finite flat. This induces by Albanese and Picard functoriality (by Picard functoriality, we mean pullback functoriality of Pic^0, and by Albanese functoriality we mean the covariant functoriality obtained by intertwining this with the autoduality of Pic^0):

$$\begin{aligned} T_p^* &:= \mathrm{Alb}(\pi_1^{(p)}) \circ \mathrm{Pic}^0(\pi_2^{(p)}) \\ (T_p)_* &:= \mathrm{Alb}(\pi_2^{(p)}) \circ \mathrm{Pic}^0(\pi_1^{(p)}), \end{aligned} \tag{5}$$

Although these are *a priori* only endomorphisms of $\mathrm{Pic}^0_{X_1(N)_{/\mathbb{Z}[1/Np]}}$, they can be extended uniquely to $\mathrm{Pic}^0_{X_1(N)_{/\mathbb{Z}[1/N]}}$ by the theory of Deligne-Rapoport [DR73] (or Katz-Mazur [KM85]) integral models. In particular we can make sense of their reductions mod p and we obtain

$$(T_p)_* \in \mathrm{End}(\mathrm{Pic}^0_{X_1(N)_{/\mathbb{F}_p}}). \tag{6}$$

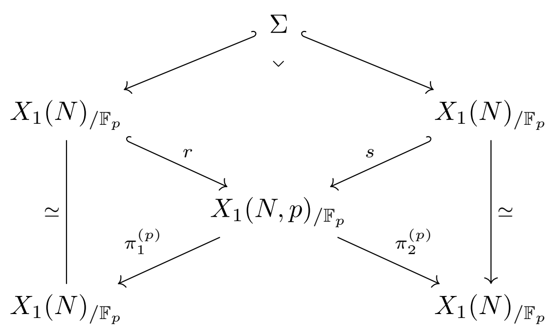
We can also define a diamond operator $\langle q \rangle_*$. Since $\mathrm{Pic}^0_{X_1(N)_{/\mathbb{F}_p}}$ is an abelian variety in characteristic p we have a Frobenius isogeny F. The first main result is

Theorem 2.1 (Eichler-Shimura congruence relation.). *One has*

$$F + \langle p \rangle_* F^\vee = (T_p)_* \tag{7}$$

in $\mathrm{End}(\mathrm{Pic}^0_{X_1(N)_{/\mathbb{F}_p}})$.

Proof sketch following [Ste]. We can use the Deligne-Rapoport (or Katz-Mazur) integral model of $X_1(N,p)$ to define its reduction $X_1(N,p)_{/\mathbb{F}_p}$. This can be viewed as two copies of $X_1(N)_{/\mathbb{F}_p}$ glued along the *supersingular locus* Σ, which is a finite set of points. (Draw the picture here). We have a diagram

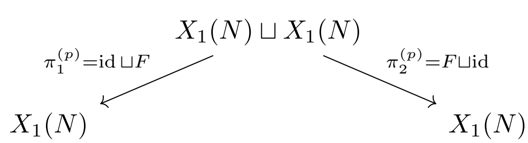


(8)

in which we have taken some liberties, since the Hecke correspondence does not literally make sense on the special fiber. The maps r and s are defined by

$$\begin{aligned} r &: (E, P) \mapsto (E, P, \ker F_E) \\ s &: (E, P) \mapsto (E^{(p)}, P, \ker V_E), \end{aligned} \tag{9}$$

where $F_E : E \leftrightarrows E^{(p)} : V_E$ are the Frobenius and Verschiebung isogenies. If one accepts the formulas for $\pi_1^{(p)}$ and $\pi_2^{(p)}$ then we see that $\pi_1^{(p)} r = \mathrm{id} = \pi_2^{(p)} s$ and $\pi_1^{(p)} s = F = \pi_2^{(p)} r$. Away from the supersingular locus, $X_1(N,p)$ looks like $X_1(N) \sqcup X_1(N)$, and we see that the Hecke correspondence looks like



(10)

Since the complement of the supersingular locus is Zariski dense, the result follows. $\square$

3 Galois representation associated to the modular curve

Let $\ell \nmid N$ be a prime. The integral and rational Tate modules

$$\begin{aligned} T_\ell(\mathrm{Pic}^0_{X_1(N)_{/\mathbb{Z}[1/N]}}) &:= \varprojlim_n \mathrm{Pic}^0_{X_1(N)_{/\mathbb{Z}[1/N]}}[\ell^n](\overline{\mathbb{Q}}) \\ V_\ell(\mathrm{Pic}^0_{X_1(N)_{/\mathbb{Z}[1/N]}}) &:= T_\ell(\mathrm{Pic}^0_{X_1(N)_{/\mathbb{Z}[1/N]}}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell. \end{aligned} \tag{11}$$

are $G_{\mathbb{Q}}$-representations of rank (resp. dimension) $2g$, where g is the genus of $X_1(N)$. Let us call the latter $\varrho_{N,\ell} : G_{\mathbb{Q}} \to GL_{2g}(\mathbb{Q}_\ell)$. Since $X_1(N)$ has good reduction at all $p \nmid N$, the Néron-Ogg-Shafarevich theorem tells us that $\varrho_{N,\ell}$ is unramified at all $p \nmid N\ell$ and hence for such p we can contemplate the endomorphism $\varrho_{N,\ell}(\mathrm{Frob}_p)$ induced by an arithmetic Frobenius at p. In fact, one has Galois and Hecke-equivariant isomorphisms

$$\begin{array}{ccccc} V_\ell(\mathrm{Pic}^0_{X_1(N)_{\mathbb{Q}}}) & \xleftarrow{\simeq} & V_\ell(\mathrm{Pic}^0_{X_1(N)_{/\mathbb{Z}[1/N]}}) & \xrightarrow{\simeq} & V_\ell(\mathrm{Pic}^0_{X_1(N)_{\mathbb{F}_p}}) \\ \simeq \big| & & & & \simeq \big| \\ H^1_{\text{ét}}(X_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^\vee & & \overset{\simeq}{\rule{8em}{0.4pt}} & & H^1_{\text{ét}}(X_1(N)_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell)^\vee \end{array} \tag{12}$$

in which the vertical arrows are induced by the Kummer exact sequence together with the long exact sequence on étale cohomology and Poincaré duality.

The groups on the bottom row are Galois representations by the action of $G_{\mathbb{Q}}$ (resp. $G_{\mathbb{Q}_p}$), on the embeddings $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}$, (resp, $\mathbb{F}_p \hookrightarrow \overline{\mathbb{F}}_p$). All arrows are $G_{\mathbb{Q}_p}$-equivariant and the left arrow is $G_{\mathbb{Q}}$-equivariant. In particular, we see that:

(i) The action of Hecke operators $(T_p)_*$ commutes with $\varrho_{N,\ell}$, since the $(T_p)_*$ were defined over $\mathbb{Q}$.

(ii) The action of $\varrho_{N,\ell}(\mathrm{Frob}_p)$ is induced by the absolute Frobenius endomorphism of $X_1(N)_{\mathbb{F}_p}$, i.e., it is induced by the Frobenius isogeny F on $\mathrm{Pic}^0_{X_1(N)_{\mathbb{F}_p}}$.

Since $FF^\vee = [p]$ (as F is an isogeny of degree p), we have that

$$F^2 - (T_p)_* F + \langle p \rangle_* [p] = F^2 - F(F + \langle p \rangle_* F^\vee) + \langle p \rangle_* [p] = 0$$

in $\mathrm{End}(\mathrm{Pic}^0_{X_1(N)_{\mathbb{F}_p}})$, where we used the Eichler-Shimura congruence relation (Theorem 2.1). Then using point (ii) above and the isomorphisms in the top row of (12) we obtain the Corollary:

Corollary 3.1. *The endomorphism $\varrho_{N,\ell}(\mathrm{Frob}_p)$ satisfies*[3] *the polynomial relation*

$$X - (T_p)_* X + \langle p \rangle_* p = 0. \tag{13}$$

in $\mathrm{End}(V_\ell(\mathrm{Pic}^0_{X_1(N)_{/\mathbb{Z}[1/N]}}))$.

[3] It is possible to prove a stronger statement that this is actually the characteristic polynomial with respect to the Hecke module structure.

4 Relation to modular forms (The Eichler-Shimura isomorphism in weight 2).

Let $X_1(N)(\mathbb{C})^{\mathrm{an}} = \Gamma_1(N)\backslash\mathfrak{h}^*$ be the upper-half plane uniformization. There is a Hecke-equivariant (for the action induced by Picard and Albanese functoriality) isomorphism

$$T_\ell(\mathrm{Pic}^0_{X_1(N)_{/\mathbb{Z}[1/N]}}) \cong T_\ell(\mathrm{Pic}^0_{X_1(N)(\mathbb{C})^{\mathrm{an}}}), \tag{14}$$

Now, we have the exponential exact sequence of sheaves on $X^{\mathrm{an}} := X_1(N)(\mathbb{C})^{\mathrm{an}}$:

$$0 \to 2\pi i\underline{\mathbb{Z}} \to \mathcal{O}_{X^{\mathrm{an}}} \to \mathcal{O}^*_{X^{\mathrm{an}}} \to 1, \tag{15}$$

and hence by the long exact sequence on sheaf cohomology one obtains an isomorphism (where we have suppresed the multiplication by $2\pi i$):

$$\mathrm{Pic}^0_{X^{\mathrm{an}}} \cong H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})/H^1(X^{\mathrm{an}}, \underline{\mathbb{Z}}), \tag{16}$$

Given a correspondence $X^{\mathrm{an}} \leftarrow Y \to X^{\mathrm{an}}$ of finite maps of Riemann surfaces, one obtains an endomorphism on the left of (16) by Picard and Albanese functoriality, and an endomorphism on the right by push-pull functoriality. The isomorphism (16) matches these up. In particular the action of the Hecke operators $(T_p)_*$ and $(T_p)^*$ as defined previous on the left matches up with $(T_p)_* := \pi^{(p)}_{2,*} \circ \pi^{(p),*}_1$ (resp. $(T_p)^* := \pi^{(p)}_{1,*} \circ \pi^{(p),*}_2$) on the right, and similarly for the diamond operators $\langle q\rangle_*, \langle q\rangle^*$.

From this it follows by the universal coefficient theorem, that one has a Hecke- equivariant isomorphism

$$T_\ell(\mathrm{Pic}^0_{X^{\mathrm{an}}}) \cong H^1(X^{\mathrm{an}}, \underline{\mathbb{Z}_\ell}) = H^1(X^{\mathrm{an}}, \underline{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \tag{17}$$

for the operators $(T_p)_*, \langle q\rangle_*$ or $(T_p)^*, \langle q\rangle^*$ on both sides.

Remark 4.1. *An important point, (that we will not have time to discuss), it that the choice of an orientation on X^{an} and an ℓ-adic orientation $\zeta = (\zeta_{\ell^n})_n$ induces a Weil pairing on $V_\ell(N) := T_\ell(\mathrm{Pic}^0_{X^{\mathrm{an}}}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$:*

$$(-,-)_\ell : V_\ell(N) \otimes V_\ell(N) \to \mathbb{Q}_\ell(1), \tag{18}$$

such that, passing to adjoints with respect to $(-,-)_\ell$, exchanges the $(-)_*$ and $(-)^*$ operators. This allows us to pass between the $(-)_*$-actions in the previous section, and the $(-)^*$-actions below. In fact one can use this pairing to show that $V_\ell(N)$ is free of rank 2 as a module over the Hecke ring.

Now the Hecke module $H^1(X^{\mathrm{an}}, \underline{\mathbb{Z}})$ will turn out to be related to weight 2 cusp forms. More precisely, the Hodge decomposition gives the Eichler-Shimura isomorphism in weight 2:

$$\begin{aligned} H^1(X^{\mathrm{an}}, \underline{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^1(X^{\mathrm{an}}, \underline{\mathbb{C}}) &\cong H^0(X, \Omega^1_{X^{\mathrm{an}}}) \oplus \overline{H^0(X, \Omega^1_{X^{\mathrm{an}}})} \\ &\cong \mathcal{S}_2(\Gamma_1(N), \mathbb{C}) \oplus \overline{\mathcal{S}_2(\Gamma_1(N), \mathbb{C})}. \end{aligned} \tag{19}$$

Let T_p, $\langle q\rangle$ be the classical Hecke operators acting on $\mathcal{S}_2(\Gamma_1(N), \mathbb{C})$. We need the following compatibility. Let $\mathbb{T}_1(N)$ be the subring of $H^1(X^{\mathrm{an}}, \underline{\mathbb{Z}})$ generated by all the $(T_p)^*, \langle q\rangle^*$.

Proposition 4.2. *The isomorphism*

$$H^1(X^{\mathrm{an}}, \underline{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathcal{S}_2(\Gamma_1(N), \mathbb{C}) \oplus \overline{\mathcal{S}_2(\Gamma_1(N), \mathbb{C})}. \tag{20}$$

matches up $\langle q\rangle^* \otimes 1$ and $(T_p)^* \otimes 1$ with $\langle q\rangle \oplus \overline{\langle q\rangle}$ and $T_p \oplus \overline{T_p}$, respectively. In particular $\mathbb{T}_1(N)$ is identified with the classical Hecke ring (of endomorphisms of $\mathcal{S}_2(\Gamma_1(N), \mathbb{C})$).

It is clear that $\mathbb{T}_1(N)$ is a finite flat $\mathbb{Z}$-algebra. It is also commutative (since the classical Hecke ring is). Hence, by going-up along the ring extension $\mathbb{Z} \to \mathbb{T}_1(N)$ we see that $\mathbb{T}_1(N)$ has finitely many minimal primes $\mathfrak{p}$ (those lying above 0). One then has

$$\mathbb{T}_1(N) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \prod_{\mathfrak{p}} K_{\mathfrak{p}}. \tag{21}$$

In turn these $K_{\mathfrak{p}}$ correspond to $G_{\mathbb{Q}}$-conjugacy classes of homomorphisms $\lambda : \mathbb{T}_1(N) \otimes_{\mathbb{Z}} \mathbb{Q} \to \overline{\mathbb{Q}}$. On the other hand, by the decomposition

$$\mathcal{S}_2(\Gamma_1(N), \mathbb{C}) = \bigoplus_{\lambda} \mathcal{S}_2(\Gamma_1(N), \mathbb{C})_{\lambda}, \tag{22}$$

into Hecke eigenspaces, we see that a newform f (new at some $\Gamma_1(M)$ with $M|N$), gives rise to such a homomorphism and hence a minimal prime $\mathfrak{p}_f \subseteq \mathbb{T}_1(N)$. This has the property that $(\mathbb{T}_1/\mathfrak{p}_f) \otimes_{\mathbb{Z}} \mathbb{Q} = K_f$ is the number field generated by the $a_p(f)$ for $p \nmid N$. We can think of K_f as the $\mathbb{T}_1 \otimes_{\mathbb{Z}} \mathbb{Q}$-module where the T_p act by multiplication by $a_p(f)$ and $\langle q \rangle$ acts by multiplication by $\chi_f(q)$ (the Nebentypus character of f).

Now we let $\mathbb{T}_1(N)$ act on $\mathrm{Pic}^0_{X_1(N)_{/\mathbb{Q}}}$ by the $(-)_*$-action, and define

$$A_f := \mathrm{Pic}^0_{X_1(N)_{/\mathbb{Q}}}/\mathfrak{p}_f, \tag{23}$$

so that we have (by definition) an exact sequence of abelian varieties

$$\mathfrak{p}_f \to \mathrm{Pic}^0_{X_1(N)_{/\mathbb{Q}}} \to A_f \to 0 \tag{24}$$

Since these are abelian varieties in characteristic 0 we can appeal to Poincaré reducibility to obtain an exact sequence of Hecke modules

$$V_\ell(\mathfrak{p}_f) \to V_\ell(\mathrm{Pic}^0_{X_1(N)_{/\mathbb{Q}}}) \to V_\ell(A_f) \to 0 \tag{25}$$

in particular the Hecke action on $V_\ell(A_f)$ factors through the action of $K_f \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Since the Galois and Hecke actions commute, $V_\ell(A_f)$ is a $G_{\mathbb{Q}}$-representation. Lastly, we choose a place $\mathfrak{l}|\ell$ of K_f and restrict to the corresponding factor $K_{f,\mathfrak{l}}$ of $K_f \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. We arrive at a $G_{\mathbb{Q}}$-representation with $K_{f,\mathfrak{l}}$-coefficients, which we call $\varrho_{N,\mathfrak{l},f}$. We have now explained everything in the following Theorem besides the 2-dimensionality.

Theorem 4.3. *There exists a continuous representation $\varrho_{N,\mathfrak{l},f} : G_{\mathbb{Q}} \to GL_2(K_{f,\mathfrak{l}})$ such that for all $p \nmid N\ell$, $\varrho_{N,\mathfrak{l},f}$ is unramified and the endomorphism $\varrho_{N,\mathfrak{l},f}(\mathrm{Frob}_p)$ satisfies the polynomial relation*

$$X^2 - a_p(f)X + p\chi_f(p) = 0. \tag{26}$$

To explain the two-dimensionality, we note that, after choice of an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, we have, by (14), (17) and (19), a Hecke-equivariant isomorphism of vector spaces

$$V_\ell(\mathrm{Pic}^0_{X_1(N)_{/\mathbb{Q}}}) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell \cong H^1(X^{\mathrm{an}}, \underline{\mathbb{C}}) \cong \mathcal{S}_2(\Gamma_1(N), \mathbb{C}) \oplus \overline{\mathcal{S}_2(\Gamma_1(N), \mathbb{C})}. \tag{27}$$

Hence, passing to f-isotypic components and using the multiplicity-one theorem for GL_2, we see that $V_\ell(A_f)$ has rank 2 as a $K_f \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-module, and the claim follows.

References

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