Eichler-Shimura isomorphism

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3rd November 2022

Abstract

These are notes for a seminar on Galois representations and modularity given in November 2022.

Contents

1 Preliminaries

Let $\mathfrak{H}$ denote the complex upper half plane and let $\mathfrak{H}^* = \mathfrak{H} \cup (\mathbb{Q} \cup \{\infty\})$ be the extended upper half plane obtained by adding in the cusps. Then $\mathfrak{H}, \mathfrak{H}^*$ have an action of $\mathrm{SL}_2(\mathbb{Z})$ by fractional linear transformations. Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. The quotients

$$Y := \Gamma \backslash \mathfrak{H} \quad X := \Gamma \backslash \mathfrak{H}^*, \tag{1}$$

have a natural complex structure under which X becomes a compact Riemann surface. A modular form of weight k and level Γ is a holomorphic function on $\mathfrak{H}$ satisfying a boundedness condition at the cusps and the transformation rule

$$f|_k\gamma(\tau) := j(\gamma,\tau)^{-k} f(\gamma(\tau)) = f(\tau), \text{ for all } \gamma \in \Gamma, \tag{2}$$

where, for $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$, $j(\gamma,\tau) := (c\tau + d)$ is the "factor of automorphy". They form a vector space $\mathcal{M}_k(\Gamma)$. If f vanishes at the cusps it is called a cusp form and the subspace of such is denoted $\mathcal{S}_k(\Gamma)$. We can form the line bundle $\Gamma\backslash(\mathfrak{H} \times \mathbb{C}) \xrightarrow{p} Y$, with the natural projection p from the first factor, where Γ acts on $\mathfrak{H}\times\mathbb{C}$ by $\gamma\cdot(\tau, z) = (\gamma(\tau), j(\gamma,\tau)^k z)$. This extends to a line bundle ω^k over X. For f a holomorphic function on $\mathfrak{H}$, the condition that $\tau \mapsto (\tau, f(\tau))$ is a holomorphic section of p is equivalent to the rule (2), and boundedness of f at the cusps is equivalent to this section extending to X. Therefore we identify

$$\mathcal{M}_k(\Gamma) = H^0(X, \omega^k), \quad \mathcal{S}_k(\Gamma) = H^0(X, \omega^k(-D)), \tag{3}$$

where $D = X - Y$ is the divisor defined by the cusps. There are $\mathbb{Q}$-schemes Y_Γ and X_Γ, where Y_Γ is affine, smooth, and identified with an open subscheme of the proper X_Γ, such that

$$Y = Y_\Gamma(\mathbb{C})^{\mathrm{an}} \quad \text{and} \quad X = X_\Gamma(\mathbb{C})^{\mathrm{an}}, \tag{4}$$

where, if the level of Γ is ≥ 3, Y_Γ is the fine moduli scheme representing the moduli functor (on $\mathsf{Sch}/\mathbb{Q}$),

$$Y_\Gamma(S) = \{\text{elliptic schemes } E/S/\mathbb{Q} \text{ with level } \Gamma \text{ structure}\}/\sim, \tag{5}$$

similarly X_Γ represents a moduli of generalised elliptic curves with level Γ structure. (The identification on $Y = Y_\Gamma(\mathbb{C})^{\mathrm{an}}$ is by sending an elliptic curve to its period.) By Yoneda, there then exists a universal elliptic curve with level Γ structure $\mathcal{E}_\Gamma$ over Y_Γ. Henceforth we shall ignore all discussion of cusps and abusively refer to $\mathcal{E}_\Gamma$ over X_Γ.

2 Eichler-Shimura isomorphism

Let $\mathcal{E}_\Gamma \xrightarrow{\pi} X_\Gamma$ be the structure map. It is proper. Then $\omega := \pi_*\Omega^1_{\mathcal{E}/X}$ is a line bundle over X_Γ such that the sheaf induced by $\omega^{\otimes k}$ on $X = X_\Gamma(\mathbb{C})^{\mathrm{an}}$ agrees with ω^k introduced previously. Therefore we have an "algebraic definition" of modular forms

$$\mathcal{M}_k(\Gamma) := H^0(X_\Gamma, \omega^{\otimes k}). \tag{6}$$

The relative de Rham cohomology $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X)$ is equipped with a decreasing Hodge filtration

$$0 \to \omega \to \mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X) \to \omega^{-1} \to 0 \tag{7}$$

and Gauss-Manin connection $\nabla : \mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X) \to \mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X) \otimes \Omega^1_{X_\Gamma}$ satisfying Griffiths transversality, which, in this situation, amounts to the map

$$\nabla : \omega = \mathrm{gr}^1\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X) \to \mathrm{gr}^0\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X) \otimes \Omega^1_{X_\Gamma} = \omega^{-1} \otimes \Omega^1_{X_\Gamma}, \tag{8}$$

being well-defined. This is in fact an isomorphism, due to the Kodaira-Spencer isomorphism:

$$\omega^{\otimes 2} \cong \Omega^1_{X_\Gamma}, \tag{9}$$

One can see this on the Riemann surface $X = \Gamma\backslash\mathfrak{H}^*$ as the map given locally by $\Omega^1_X(U) \ni f(\tau)d\tau \mapsto f(\tau)$, which is then a weight 2 modular form because of the rule $d\gamma(\tau) = j(\gamma,\tau)^{-2}d\tau$, i.e., a section of ω^2. Let $\underline{\mathbb{Z}}$ be the constant local system on $\mathcal{E}_\Gamma$. Then $R^1\pi_*\underline{\mathbb{Z}}$ is (non-canonically) isomorphic to the locally constant sheaf $\underline{\mathbb{Z}}^2$ on X_Γ. Therefore there are isomorphisms

$$\begin{aligned} H^1(\Gamma, \mathrm{Sym}^{k-2}\mathbb{Z}^2) \otimes \mathbb{C} &\cong H^1_{\mathrm{Betti}}(\mathfrak{H}^*/\Gamma, \mathrm{Sym}^{k-2}\mathbb{Z}^2) \otimes \mathbb{C} \\ &= H^1(X_\Gamma(\mathbb{C})^{\mathrm{an}}, \mathrm{Sym}^{k-2}R^1\pi_*\underline{\mathbb{Z}}) \otimes \mathbb{C} \\ &= H^1(X_\Gamma(\mathbb{C})^{\mathrm{an}}, \mathrm{Sym}^{k-2}R^1\pi_*\underline{\mathbb{C}}_{\mathcal{E}}). \end{aligned} \tag{10}$$

Recall that the Riemann-Hilbert correspondence, for a smooth variety $Z/\mathbb{C}$ is

$$\{\mathbb{C}-\text{local systems } \mathbb{L} \text{ on } Z(\mathbb{C})^{\mathrm{an}}\} \leftrightarrow \left\{\begin{matrix} \mathcal{O}_Z-\text{modules with} \\ \text{integrable connection } (\mathcal{M},\nabla) \end{matrix}\right\}, \tag{11}$$

under which $\underline{\mathbb{C}}_Z$ corresponds to $\mathcal{O}_Z$. Moreover if $\mathbb{L}$ corresponds to $(\mathcal{M},\nabla)$ one has $H^i(Z(\mathbb{C})^{\mathrm{an}},\mathbb{L}) = H^i_{\mathrm{dR}}(Z,\mathcal{M})$. It extends to a derived equivalence between perverse sheaves and regular holonomic $\mathcal{D}$-modules, compatible with the six functors on both sides, such that we recover (11) by taking cohomology. In our situation this implies $R^1\pi_*\mathbb{C}$ corresponds to $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X)$ since this is the first cohomology of the $\mathcal{D}$-module pushforward

$\int_\pi \mathcal{O}_{\mathcal{E}}$. See [HTT08, Theorem 7.1.1]. Then, taking Sym^{k-2}, $\mathrm{Sym}^{k-2}R^1\pi_*\mathbb{C}$ corresponds to $\mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X)$. Therefore

$$H^1(X_\Gamma(\mathbb{C})^{\mathrm{an}}, \mathrm{Sym}^{k-2}R^1\pi_*\underline{\mathbb{C}}_{\mathcal{E}}) \cong H^1_{\mathrm{dR}}(X_{\Gamma,\mathbb{C}}, \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X)). \tag{12}$$

From the convergence of the Hodge-de Rham spectral sequence,

$$E_1^{p,q} = H^p(X_{\Gamma,\mathbb{C}}, \Omega^q_{X_\Gamma} \otimes \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X)) \Rightarrow H^{p+q}_{\mathrm{dR}}(X_{\Gamma,\mathbb{C}}, \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X)), \tag{13}$$

we have an exact sequence of low degree terms

$$\begin{aligned} 0 \to H^0(X_{\Gamma,\mathbb{C}}, \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X)) &\to H^0(X_{\Gamma,\mathbb{C}}, \Omega^1_{X_\Gamma} \otimes \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X)) \\ &\to H^1_{\mathrm{dR}}(X_{\Gamma,\mathbb{C}}, \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X)) \\ &\to H^1(X_{\Gamma,\mathbb{C}}, \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X)) \\ &\to H^1(X_{\Gamma,\mathbb{C}}, \Omega^1_{X_\Gamma} \otimes \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X)) \to 0 \end{aligned} \tag{14}$$

Now examine the first two terms in the sequence. The last two terms will be essentially the same by Serre duality. Consider

$$\begin{array}{ccc} H^0(X_{\Gamma,\mathbb{C}}, \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X)) & \xrightarrow{\nabla_*} & H^0(X_{\Gamma,\mathbb{C}}, \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X) \otimes \Omega^1_{X_\Gamma}) \\ & & \uparrow \\ & & H^0(X_{\Gamma,\mathbb{C}}, \omega^{k-2} \otimes \Omega^1_{X_\Gamma}) \end{array} \tag{15}$$

where the vertical map is induced by the Hodge filtration (7) on $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X)$ and the horizontal map ∇_* is induced by the Gauss-Manin connection ∇. We claim that

$$\mathcal{S}_k(\Gamma) \cong H^0(X_{\Gamma,\mathbb{C}}, \omega^{k-2} \otimes \Omega^1_{X_\Gamma}) \text{ maps isomorphically onto } \mathrm{coker}\nabla_*, \tag{16}$$

where on the left we used the Kodaira-Spencer isomorphism. Indeed, the Hodge filtration on $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X)$ induces one on $\mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X)$, with

$$\mathrm{gr}^p\mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X) \cong \omega^{2-k+2p}, \tag{17}$$

therefore, again by Kodaira-Spencer,

$$\mathrm{gr}^\bullet\nabla \text{ maps } \mathrm{gr}^p\mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}} \text{ isomorphically onto } \mathrm{gr}^{p-1}\mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}} \otimes \Omega^1_{X_\Gamma}, \tag{18}$$

for $p \geq k-2$. It follows that

$$\mathrm{coker}\ \mathrm{gr}^\bullet\nabla_* \cong H^0(X_{\Gamma,\mathbb{C}}, \omega^{k-2} \otimes \Omega^1_{X_\Gamma})[2-k] = \mathrm{gr}^\bullet H^0(X_{\Gamma,\mathbb{C}}, \omega^{k-2} \otimes \Omega^1_{X_\Gamma}), \tag{19}$$

since on the right hand side the filtration only jumps in degree $k-2$. The claim now follows since $\mathrm{gr}^\bullet$ is a conservative functor. Putting this all together, (and doing the same, after Serre duality, for the last two terms in (14)), we have obtained a natural short exact sequence

$$0 \to \mathcal{S}_k(\Gamma) \xrightarrow{\delta} H^1_{\mathrm{dR}}(X_{\Gamma,\mathbb{C}}, \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X)) \to \mathcal{S}_k(\Gamma)^\vee \to 0, \tag{20}$$

The de Rham's theorem comparison

$$H^1_{\mathrm{Betti}}(X, \mathrm{Sym}^{k-2}\underline{\mathbb{Z}}^2) \otimes \mathbb{C} \cong H^1_{\mathrm{dR}}(X_{\Gamma,\mathbb{C}}, \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/X)) \tag{21}$$

endows the middle term of (20) with a complex conjugation ι, from $\mathrm{id}\otimes\overline{(-)}$ on the left side, under which (20) becomes a Hodge filtration in weights $(k-1,0),(0,k-1)$. In particular $\overline{\delta}:\overline{\mathcal{S}_k(\Gamma)}\cong\mathcal{S}_k(\Gamma)^\vee\to H^1_{\mathrm{dR}}(\dots)$ gives a splitting of the quotient map in (20) and hence, combining with , an isomorphism

$$\beta:\mathcal{S}_k(\Gamma)\oplus\overline{\mathcal{S}_k(\Gamma)}\xrightarrow{\sim}H^1_{\mathrm{Betti}}(X,\mathrm{Sym}^{k-2}\mathbb{Z}^2)\otimes\mathbb{C}. \tag{22}$$

This is known as the Eichler-Shimura isomorphism.

3 Hecke operators

Let $\Gamma\leq\mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup, let $\alpha\in\mathrm{GL}_2(\mathbb{Q})$ and set

$$\Gamma^\alpha=\alpha\Gamma\alpha^{-1}\cap\Gamma,\quad\Gamma_\alpha=\alpha^{-1}\Gamma\alpha\cap\Gamma, \tag{23}$$

these are again congruence subgroups. The isomorphism $[\alpha]:\Gamma_\alpha\to\Gamma^\alpha:\gamma\mapsto\alpha\gamma\alpha^{-1}$ induces an isomorphism $\Gamma_\alpha\backslash\mathfrak{H}^*\to\Gamma^\alpha\backslash\mathfrak{H}^*$. The inclusions $\Gamma_\alpha,\Gamma^\alpha\subseteq\Gamma$ induce finite unramified coverings of Riemann surfaces $\Gamma_\alpha\backslash\mathfrak{H}^*,\Gamma^\alpha\backslash\mathfrak{H}^*\to\Gamma\backslash\mathfrak{H}^*$. Therefore, we obtain a trace map on cohomology, and the composite

$$H^1_{\mathrm{Betti}}(X,A)\xrightarrow{\mathrm{Res}}H^1_{\mathrm{Betti}}(\Gamma^\alpha\backslash\mathfrak{H}^*,A)\xrightarrow{[\alpha]^*}H^1_{\mathrm{Betti}}(\Gamma_\alpha\backslash\mathfrak{H}^*,A)\xrightarrow{\mathrm{tr}}H^1_{\mathrm{Betti}}(X,A) \tag{24}$$

is an A-linear map depending only on the double coset $\Gamma\alpha\Gamma$; here A is an arbitrary abelian group. If A is a $\mathrm{GL}_2(\mathbb{Q})$-equivariant local system we include an isomorphism $\alpha_*A\cong A$ in the composite (24). This operator is notated $[\Gamma\alpha\Gamma]$. For $f\in\mathcal{S}_k(\Gamma)$ or $\mathcal{M}_k(\Gamma)$ we define $f[\Gamma\alpha\Gamma]_k:=\sum_j f|_k\beta_j$ where $\Gamma\alpha\Gamma=\bigsqcup_j\Gamma\beta_j$ is a system of coset representatives.

Since $\Gamma_1(N)\trianglelefteq\Gamma_0(N)$ the quotient $\Gamma_0(N)/\Gamma_1(N)=(\mathbb{Z}/N\mathbb{Z})^\times$ acts on $\Gamma_1(N)$ by conjugation. Hence in the above discussion, taking $\alpha\in\Gamma_0(N)$ and $\Gamma=\Gamma_1(N)$, induces an action of $(\mathbb{Z}/N\mathbb{Z})^\times$ on $H^1_{\mathrm{Betti}}(X,A)$ and $\mathcal{S}_k(\Gamma_1(N))$. The operator induced by $d\in(\mathbb{Z}/N\mathbb{Z})^\times$ is denoted $\langle d\rangle_k$. Also, if one takes $\alpha=\begin{pmatrix}1&0\\0&p\end{pmatrix}$ (and $\Gamma=\Gamma_1(N)$), the resulting operators on $H^1_{\mathrm{Betti}}(X,A)$ and $\mathcal{S}_k(\Gamma_1(N))$ are denoted T_p. Let $\mathbb{T}_k$ denote the subring of $\mathrm{End}(\mathcal{S}_k(\Gamma_1(N)))$ generated by $\{\langle q\rangle_k,T_p:q,p\nmid N\}$. Let[1] $R(\Gamma_1(N))$ be the subring of $\mathrm{End}(H^1_{\mathrm{Betti}}(X,\mathrm{Sym}^{k-2}\mathbb{Z}^2))$ generated by $\{\langle q\rangle_k,T_p:q,p\nmid N\}$. By definition, we see that $R(\Gamma_1(N))$ is a finite $\mathbb{Z}$-module. Moreover, the Eichler-Shimura isomorphism is equivariant for these Hecke actions in the sense that, via (22), the action of $R(\Gamma_1(N))$ on $H^1_{\mathrm{Betti}}(X,\mathrm{Sym}^{k-2}\mathbb{Z}^2)$ induces an action on $\mathcal{S}_k(\Gamma_1(N))$ which agrees with $\mathbb{T}_k$. It follows that

Corollary 3.1. *$\mathbb{T}_k$ is a finite free $\mathbb{Z}$-module.*

The elements of $\mathbb{T}_k$ are commuting linear operators on $\mathcal{S}_k(\Gamma_1(N))$, and normal with respect to the Petersson inner product. Therefore

$$\mathcal{S}_k(\Gamma_1(N))=\bigoplus_\lambda\mathcal{S}_k(\Gamma_1(N))_\lambda, \tag{25}$$

over all systems of eigenvalues $\lambda:\mathbb{T}_k\to\mathbb{C}$. Therefore, choosing a simultaneous basis of eigenforms simultaneously diagonalises the operators $\mathbb{T}_k$ and defines an algebra isomorphism $\mathbb{T}_k\otimes_{\mathbb{Z}}\mathbb{C}\cong\mathbb{C}^{\#\{\text{distinct }\lambda\}}$. Since $\mathbb{T}_k$ is free, $\mathbb{T}_k\hookrightarrow\mathbb{T}_k\otimes\mathbb{C}$ and so it is reduced. Corollary

[1] This is my (made-up) notation.

3.1 implies that $\mathbb{Z} \to \mathbb{T}_k$ is an integral extension and since $\mathbb{Z}$ is an integrally closed domain this extension satisfies going-up and going-down. In particular the minimal primes of $\mathbb{T}_k$ are precisely the finitely many $\mathfrak{p} \subseteq \mathbb{T}_k$ lying above 0. Since $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Q}$ is a finite $\mathbb{Q}$-algebra, it has a canonical decomposition

$$\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Q} \cong \prod_{\mathfrak{p}} K_{\mathfrak{p}}, \tag{26}$$

where the fields $K_{\mathfrak{p}}$ are the localisations of $\mathbb{T}_k$ at these minimal primes. These primes can be identified with the kernels of homorphisms $\lambda : \mathbb{T}_k \to \overline{\mathbb{Q}}$, which determines λ up to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugacy[2]. In turn we can choose a unique normalised newform $f \in \mathcal{S}_k(\Gamma_1(M))$ (for some $M|N$), for λ. In summary the following finite sets are in natural bijection:

- Minimal primes of $\mathbb{T}_k$.
- Maximal ideals of $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Q}$.
- $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugacy classes of normalised newforms in $\mathcal{S}_k(\Gamma_1(M))$, where $M|N$.

References

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki. *$\mathcal{D}$-modules, perverse sheaves, and representation theory. Translated from the Japanese by Kiyoshi Takeuchi*, volume 236 of *Prog. Math.* Basel: Birkhäuser, expanded edition edition, 2008. ISSN: 0743-1643.

[2]If X/k is a scheme of locally finite type over a field then its closed points are in bijection with $X(\overline{k})/\mathrm{Gal}(\overline{k}/k)$.