Some perspectives on differential operators in algebraic geometry

Arun Soor

May 11, 2023

Abstract

These are notes for a study group on Geometric Langlands in Oxford in May 2023.

Contents

1 Introduction

Let $M/\mathbb{C}$ be a complex manifold and let $E \to M$ be a rank n vector bundle. A connection on M is a $\mathbb{C}$-linear map $\Gamma(E) \to \Gamma(T^*M \otimes E)$ satisfying the Leibniz rule $\nabla(fs) = df \otimes s + f\nabla(s)$, for all $f \in \Gamma(M), s \in \Gamma(E)$. By evaluating ∇ at vector fields ∂, it can equivalently be viewed as a map $\Gamma(TM) \to \mathrm{End}(E) : \partial \mapsto \nabla_\partial$.

Given $x, y \in M$, let γ be a smooth path with $\gamma(0) = x, \gamma(1) = y$, then $\dot\gamma$ is a vector field along γ. Let $e_x \in E_x$, then by the existence of solutions to linear ODEs with given initial conditions (Picard-Lindelöf theorem), there exists a unique section s of E along γ such that $\nabla_{\dot\gamma} s = 0$ and $s(0) = e_x$. Set $e_y = s(1)$, then by considering the reversed path we have determined an isomorphism $\Gamma(\gamma)_x^y : E_x \xrightarrow{\sim} E_y$ which, if ∇ is *flat*, can be shown to depend only on the path homotopy class of γ. This is known as parallel transport along γ. In particular the connection has given us a canonical identification of “nearby” fibers of E, basically since $M \cong \mathbb{C}^{\dim M}$ locally, which is simply-connected.

In this talk we will present a generalisation of vector bundles with *flat* connection ($\mathcal{D}$-modules), and an algebraic incarnation of “identifying nearby fibers” (crystals), and relate them.

2 $\mathcal{D}$-modules

Let $X/\mathbb{C}$ be a smooth scheme and let $\mathcal{D}_X \subset \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$ be the subsheaf of algebras generated by $\mathcal{O}_X$ and $\mathcal{T}_X = \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$. A $\mathcal{D}_X$-module is a $\mathcal{D}_X$-module object $M \in \mathrm{QCoh}(X)$, extending the $\mathcal{O}_X$-module structure. This is the same[1] as giving a connection $\nabla : M \to M \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbb{C}}$ which is flat, i.e. $\nabla \wedge \nabla : M \to M \otimes_{\mathcal{O}_X} \Omega^2_{X/\mathbb{C}}$. Therefore $\mathcal{D}_X$-modules generalise vector bundles with connection by removing the finite locally free hypothesis.

2.1 Differential operators

For $M, N \in \mathrm{QCoh}(X)$ recursively define $\mathcal{D}iff^0(M,N) := \mathcal{H}om_{\mathcal{O}_X}(M,N)$ and

$$\mathcal{D}iff^{n+1}(M,N) := \{D \in \mathcal{H}om_{\mathbb{C}}(M,N) : fD - Df \in \mathcal{D}iff^n(M,N) \text{ for all } f \in \mathcal{O}_X\}, \tag{1}$$

and set $\mathcal{D}iff(M,N) := \bigcup_{n\geq 0} \mathcal{D}iff^n(M,N)$. For example $\mathcal{D}_X = \mathcal{D}iff(\mathcal{O}_X, \mathcal{O}_X)$. It is clear that $\mathcal{D}iff(M,N)$ is filtered, and in fact if M, N are locally free then $\operatorname{gr} \mathcal{D}iff(M,N) \cong \mathrm{Hom}_{\mathcal{O}_X}(M,N) \otimes_{\mathcal{O}_X} \mathrm{Sym}^\bullet \mathcal{T}_X$. Thus if $\mathcal{L}$ is a line bundle then $\mathcal{D}_X(\mathcal{L},\mathcal{L}) := \mathcal{D}iff(\mathcal{L},\mathcal{L}) \cong \mathrm{Sym}^\bullet \mathcal{T}_X$ and this is in fact an isomorphism of Poisson algebras[2] (recall that for any filtered graded-commutative ring $\mathcal{D}$, $\operatorname{gr}\mathcal{D}$ has the canonical structure of a Poisson algebra).

Definition 2.1. *[Gin98, Definition 2.2.1] A TDO is a positively filtered sheaf $\mathcal{D}$ of $\mathbb{C}$-algebras together with an isomorphism $\operatorname{gr}\mathcal{D} \cong \mathrm{Sym}^\bullet \mathcal{T}_X$ of Poisson algebras.*

The category $\mathsf{Mod}_{\mathcal{D}_X}$ is abelian and closed symmetric monoidal with respect to $\otimes_{\mathcal{O}_X}$, $\mathcal{H}om_{\mathcal{O}_X}(-,-)$. Unfortunately, given a smooth morphism $f : X \to Y$ of smooth schemes there is no obvious morphism of ringed spaces $(X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ extending this. This makes push/pull of $\mathcal{D}$-modules difficult to define. In fact, f_* only exists at the derived level. The next section is intended to give a more intuitive description of these functors.

2.2 DG-modules over the de Rham complex

The main reference for this section is [Kap91], see also [BD, §7.2, 7.3]. The content of this section ($\mathcal{D}$-Ω duality) can be seen as an instance of Koszul duality [Pos11, Appendix B].

Let $X/\mathbb{C}$ be a smooth quasi-projective variety. Recall that a DG-algebra is a graded algebra A with degree 1 differential d satisfying $d \circ d = 0$ and the graded Leibnitz rule $d(ab) = (da)\cdot b + (-1)^{\deg a} a \cdot (db)$ (for homogeneous a, b), i.e., a monoid object in chain complexes. For example the de Rham complex $\Omega^\bullet_X$ is a sheaf of DG-algebras on X_{Zar}, we define an $\Omega^\bullet_X$-module as a module object for this in $\mathrm{Ch}(\mathrm{QCoh}(X))$. A morphism of such is just an $\mathcal{O}_X$-linear map of complexes and we denote the category of such by $\mathcal{M}_{qc}(\Omega^\bullet_X)$. This is nothing but the full subcategory of $M^\bullet \in \mathrm{Ch}(\mathrm{QCoh}(X))$ where we require $d \in \mathcal{D}iff^1(M^i, M^{i+1})$ for all i.

We can also consider graded left modules (without differential) over the graded algebra $\Omega^\bullet_X$, which we call $\Omega^\#_X$-modules.

$\mathcal{M}^b_c(\Omega^\bullet_X)$ shall denote the category of $\Omega^\bullet_X$-modules $M^\bullet$, such that $M^\bullet$ is a bounded complex of coherent $\mathcal{O}_X$-modules.

If $s : M^\bullet \to N^\bullet[1]$ is a morphism of $\Omega^\#_X$-modules then $f = d_N s + s d_M$ is a morphism of $\Omega^\bullet_X$-modules which we call "homotopic to 0" and we form the homotopy category $\mathbf{K}^b_c(\Omega^\bullet_X)$

[1] The first condition just says that $\mathcal{T}_X$ acts by derivations and the flatness says that the components of $\mathcal{T}_X$ commute with each other.

[2] A Poisson algebra is a commutative algebra with Lie bracket $\{\cdot,\cdot\}$ which is a bi-derivation.

as a quotient of $\mathcal{M}_c^b(\Omega_X^\bullet)$. A map $f : M^\bullet \to N^\bullet$ in $\mathcal{M}_c^b(\Omega_X^\bullet)$ is called a quasi-isomorphism if $f_{\text{an}} : M^\bullet_{\text{an}} \to N^\bullet_{\text{an}}$ is a quasi-isomorphism of complexes of sheaves on X_{an}; localising, we form $D_c^b(\Omega_X^\bullet)$. Given $M^\bullet \in \mathcal{M}(\Omega_X^\bullet)$ consider the complex

$$DR^{-1}(M^\bullet) := \left[\cdots \to M^0 \otimes_{\mathcal{O}_X} \mathcal{D}_X \xrightarrow{\delta} M^1 \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \cdots\right] \tag{2}$$

where δ is defined by

$$\begin{array}{ccc} M^i \otimes_{\mathcal{O}_X} \mathcal{D}_X & \xrightarrow{\delta} & M^{i+1} \otimes_{\mathcal{O}_X} \mathcal{D}_X \\ \Big| \cong & & \Big| \cong \\ \mathcal{D}iff(\mathcal{O}_X, M^i) & \xrightarrow{d_M \circ -} & \mathcal{D}iff(\mathcal{O}_X, M^{i+1}) \end{array} \tag{3}$$

This extends to an exact functor $\widetilde{DR}^{-1} : D_c^b(\Omega_X^\bullet) \to D_c^b(\mathcal{D}_X)$. We define also the functor $\widetilde{DR} : D_c^b(\mathcal{D}_X) \to D_c^b(\Omega_X^\bullet) : N^\bullet \mapsto N^\bullet \otimes^{\mathbb{L}}_{\mathcal{D}_X} \mathcal{O}_X$, it is well-defined by the existence of the Spencer resolution.

Theorem 2.2. *[Kap91, Theorem 1.4],[Sai89, Proposition 1.2] The functors $\widetilde{DR}^{-1}$, $\widetilde{DR}$ are mutual quasi-inverses giving an equivalence of categories $D_c^b(\Omega_X^\bullet) \cong D_c^b(\mathcal{D}_X)$.*

Thus $D_c^b(\Omega_X^\bullet)$ is a "direct" definition of $D_c^b(\mathcal{D}_X)$. For a smooth morphism $f : X \to Y$ of smooth varieties over $\mathbb{C}$ put f_*, f^{-1} for the sheaf-theoretic direct/inverse images, then f induces a morphism of DG-ringed spaces $(X, \Omega_X^\bullet) \to (Y, \Omega_Y^\bullet)$, i.e, we have a DG-algebra map $\Omega_Y^\bullet \to f_*\Omega_X^\bullet$, equivalently $f^{-1}\Omega_Y^\bullet \to \Omega_X^\bullet$. Thus we can define push/pull on $\mathcal{M}_{qc}(\Omega^\bullet)$ in the usual way, i.e., $f_{\Omega,*}M^\bullet := f_*M^\bullet$ with the action given by restriction along $\Omega_Y^\bullet \to f_*\Omega_X^\bullet$, and $f_\Omega^* N^\bullet := \Omega_X^\bullet \otimes_{f^{-1}\Omega_Y^\bullet} f^{-1}N^\bullet$, for $M^\bullet \in \mathcal{M}_{qc}(\Omega_X^\bullet), N^\bullet \in \mathcal{M}_{qc}(\Omega_Y^\bullet)$. The pushforward $f_{\Omega,*}$ has to be "derived" to get a functor $Rf_{\Omega,*} : D_c^b(\Omega_X^\bullet) \to D_c^b(\Omega_Y^\bullet)$. By using Theorem 2.2 one can recover the usual formulas for push/pull of $\mathcal{D}$-modules.

The analogue of Theorem 2.2 for $D_{qc}^?(\mathcal{D}_X)$, $? \in \{\emptyset, +, -, b\}$ is more subtle to define. Just as above, we are always able to define an adjoint pair of functors

$$\widetilde{DR}^{-1} : \mathbf{K}_{qc}^?(\Omega_X^\bullet) \leftrightarrows \mathbf{K}_{qc}^?(\mathcal{D}_X) : \widetilde{DR} \tag{4}$$

however we must be careful about which quasi-isomorphisms to invert on the left, c.f. [BD04, §2.1.10]. One defines a *$\mathcal{D}$-quasi-isomorphism* as those ψ where $\widetilde{DR}^{-1}(\psi)$ is a quasi-isomorphism in $\mathbf{K}_{qc}^?(\mathcal{D}_X)$ and $D_{qc}^?(\Omega_X^\bullet)$ is the localisation at these, then we get the equivalence $D_{qc}^?(\Omega_X^\bullet) \cong D_{qc}^?(\mathcal{D}_X)$.

3 Crystals

As mentioned, crystals are supposed to be an algebraic incarnation of "identifying nearby fibers". The main reference for this section is [Lur09].

Let X/k be any separated scheme over any field k of characteristic 0 (not necessarily smooth), which we may view as a functor on commutative k-algebras R. For a quasi-coherent sheaf M on X and $x \in X(R)$ we have the pullback $x^*M \in \mathsf{Mod}_R$. We say $x, y \in X(R)$ are infinitesimally close if they agree in the image of $X(R) \to X(R^{\text{red}})$. Then a crystal in quasi-coherent sheaves is such an M, together with the data of isomorphisms $\alpha_{x,y} : x^*M \to y^*M$ for every pair $x, y \in X(R)$ of infinitesimally close points, compatible with base change in R and satisfying a cocycle condition (coming from the transitivity of the relation of being "infinitesimally close").

If one defines the functor $X_{\mathrm{dR}}(R) := X(R^{\mathrm{red}})$ then this is the same as the data of a quasi-coherent sheaf on X_{dR}, where for an arbitrary functor $\mathsf{CommAlg}_k \to \mathsf{Set}$ a quasi-coherent sheaf on it is defined as in [Lur09]. We think of $X(R) \to X_{\mathrm{dR}}(R)$ as giving $X_{\mathrm{dR}}(R)$ the structure of a groupoid (where we have divided by the relation of being infinitesimally close), the "infinitesimal groupoid". If X is smooth then X_{dR} is a sheaf on $\mathsf{Sch}^{\mathsf{op}}_{/k}$ and accordingly is called the de Rham stack.

Consider (c.f. [Gro68, Appendix]) the diagram

$$\widehat{(X \times X \times X)}_\Delta \overset{p_{12}}{\underset{p_{31}}{\overset{-p_{23}\to}{\rightrightarrows}}} \widehat{(X \times X)}_\Delta \overset{p_1}{\underset{p_2}{\rightrightarrows}} X \tag{5}$$

where $\widehat{(X \times X)}_\Delta$, etc, is the formal completion along the diagonal, p_{12}, p_1, etc, are the projections[3]. We claim that a crystal is the same as the data (M, φ) where $M \in \mathrm{QCoh(X)}$ and $\varphi : p_1^* M \xrightarrow{\sim} p_2^* M$ is an isomorphism which restricts to id on the diagonal and satisfies the cocycle condition $p_{23}^*(\varphi) \circ p_{12}^*(\varphi) = p_{13}^*(\varphi)$: morally speaking, X_{dR} is the coequaliser of the diagram (5).

For, to say that $x, y : \mathrm{Spec}(R) \to X$ are infinitesimally close is the same as saying that $(x, y) : \mathrm{Spec}(R^{\mathrm{red}}) \to X \times X$ factors through the diagonal, i.e., $(x, y)^* \mathcal{J} \subset \mathrm{nilrad}(R)$, where $\mathcal{J}$ is the ideal defining the diagonal; so[4]

$$(x, y)^* \mathcal{J}^{n+1} = 0 \text{ for some } n \geq 0. \tag{6}$$

As a formal scheme, $\widehat{(X \times X)}_\Delta$ is just a particular kind of ind-scheme[5], and so $\widehat{(X \times X)}_\Delta(R) = \varinjlim_n (X \times X)^n_\Delta(R)$, one then notes that (6) says exactly that $(x, y) \in (X \times X)^n_\Delta(R)$. Thus we have shown that $\widehat{(X \times X)}_\Delta$ is universal for pairs of infinitesimally close morphisms (x, y), hence to give all the data of a crystal it is sufficient to give $M \in \mathrm{QCoh}(X)$ with an isomorphism $\varphi : p_1^* M \xrightarrow{\sim} p_2^* M$ satisfying the cocycle condition.

If X is smooth this is the same as a $\mathcal{D}_X$-module. For, by the usual adjunction φ translates to a map

$$\tilde{\varphi} : M \to p_{1,*} p_2^* M = \varprojlim_n \mathcal{O}_{X \times X} / \mathcal{J}^{n+1} \otimes_{\mathcal{O}_X} M \tag{7}$$

since X is smooth we can take étale coordinates $\{x_i\}$ locally and identify $\mathcal{D}_X$ with the restricted (filtered) $\mathcal{O}_X$-dual of $\varprojlim_n \mathcal{O}_{X \times X}/\mathcal{J}^{n+1}$ by the pairing $\langle \partial^\alpha, \frac{1}{\beta!}(x' - x'')^\beta \rangle = \delta_{\alpha\beta}$ (extended bilinearly). Therefore the "coaction" (7) can be transposed to an action $\tilde{\varphi}^t : \mathcal{D}_X \otimes_{\mathcal{O}_X} M \to M$; that this extends the $\mathcal{O}_X$ action and is associative, is equivalent to $\varphi|_\Delta = \mathrm{id}$ and the cocycle condition.

3.1 $\mathcal{D}$-schemes, jets, conformal blocks

The main references for this section are [Neg09, Lur09]. We can define crystals valued in all sorts of objects. For example if S/k is a smooth scheme then we define a *crystal*

[3] This makes the pair $(X, \widehat{(X \times X)}_\Delta)$ into a formal groupoid in the sense of Simpson [Sim97, §7], who then defines X_{dR} as the stack associated to this formal groupoid; he then shows that if X is smooth then $X_{\mathrm{dR}}(R) = X(R^{\mathrm{red}})$. Here we are starting the other way round.

[4] For simplicity assume R is Noetherian...

[5] i.e., just the ones whose reduction is actually a scheme.

of schemes over S as an S-scheme $Z \xrightarrow{\pi} S$, with the following additional data: for each $R \in \mathsf{CommAlg}_k$ and each pair of infinitesimally close morphisms $x, y \in S(R)$ an isomorphism $x^*Z \xrightarrow{\sim} y^*Z$, compatible with base change in R and satisfying a cocycle condition. Here $x^*Z := Z \times_{S,x} \mathrm{Spec}(R)$.

In a manner analogous to previous, there is a relation to $\mathcal{D}_S$-modules, namely a canonical equivalence

$$\mathsf{CommMon}(\mathsf{Mod}_{\mathcal{D}_S})^{\mathsf{op}} \cong \{\text{crystals of } S\text{-schemes } \pi : Z \to S \text{ with } \pi \text{ affine}\}\,, \tag{8}$$

objects on the left are "affine" $\mathcal{D}_S$-schemes. More generally a $\mathcal{D}_S$-scheme is an S-scheme equipped with a flat connection $\mathcal{O}_Z \to \mathcal{O}_Z \otimes_{\mathcal{O}_S} \Omega^1_{S/k}$, for example $\underline{\mathrm{Spec}}_S(\mathrm{Sym}_{\mathcal{O}_S} \mathcal{M})$ for any $\mathcal{D}_S$-module $\mathcal{M}$. $\mathcal{D}_S$-schemes give a coordinate-free way of writing nonlinear differential equations. They have an obvious forgetful functor to S-schemes, which has an adjoint $\mathcal{J}$, the functor of jets. Given a commutative $\mathcal{O}_S$-algebra A with $X = \underline{\mathrm{Spec}}_S(A)$ one sets

$$\mathcal{J}X := \underline{\mathrm{Spec}}_S((\mathrm{Sym}_{\mathcal{O}_S} \mathcal{D}_X \otimes_{\mathcal{O}_S} A)/\ker(\mathrm{Sym}_{\mathcal{O}_S} A \to A)) \tag{9}$$

and this can be globalised by gluing. Given a morphism of $\mathcal{D}_S$-schemes $Y \to Z$ one defines the functor (on $\mathsf{Sch}^{\mathsf{op}}_{/k}$) of horizontal sections

$$\mathrm{HorSect}(Z, Y)(T) := \mathrm{Hom}_{\mathsf{Sch}_{\mathcal{D}_S}/Z}(Z \times T, Y), \tag{10}$$

they are "horizontal" since they are automatically $\mathcal{D}_S$-scheme maps. If X is an $S-$scheme with a map to the $\mathcal{D}_S$-scheme Z then the adjunction gives as $\mathcal{D}_S$-scheme map $\mathcal{J}X \to Z$, and unraveling the adjunctions one has

$$\mathrm{HorSect}(Z, \mathcal{J}X) = \mathrm{Sect}(Z, X), \tag{11}$$

where the functor of sections is given by $\mathrm{Sect}(Z, X)(T) := \mathrm{Hom}_{\mathsf{Sch}_S/Z}(Z \times T, X)$. Therefore one can recover X from its jet-scheme. A particular case is the functor of conformal blocks, defined by $H_\nabla(S, Y) := \mathrm{HorSect}(S, Y)$ for $Y \in \mathsf{Sch}_{\mathcal{D}_S}$, here $Y \to S$ is the structural map.

This has the following interpretation in terms of crystals. A crystal of S-schemes (equivalently $\mathcal{D}_S$-scheme) is the same as a relatively representable functor Z over the de Rham stack S_{dR}. The forgetful functor from crystals of S-schemes to S-schemes is given by pullback along the "tautological" 2-morphism $p_{\mathrm{dR},S} : S \to S_{\mathrm{dR}}$, i.e., $p^*_{\mathrm{dR},S} = - \times_{S_{\mathrm{dR}}} S$, and the jet-functor is given by pushforward $p_{\mathrm{dR},S,*}$, i.e., Weil restriction [Sta, Tag 05Y8]. Given crystals of S-schemes Y, Z with a map $Y \to Z$ over S_{dR}, the functor of horizontal sections is given by pushforward (Weil restriction) of functors along $Z \to \mathrm{pt}$, and the conformal block functor is the particular case when we take $Z = S$ and the structural map $Y \to S$. Therefore we see that H_∇ is adjoint to the functor taking a scheme T to the constant $\mathcal{D}_S$-scheme $S \times T$, i.e.,

$$\mathrm{Hom}_{\mathsf{Sch}/k}(H_\nabla(S, Y), X) \cong \mathrm{Hom}_{\mathsf{Sch}_{\mathcal{D}_S}}(Y, X \times S), \tag{12}$$

for any $\mathcal{D}_S$-scheme Y and $X \in \mathsf{Sch}/k$.

References

[BD] Alexander Beĭlinson and Vladimir Drinfel'd. Quantization of Hitchin's integrable system and Hecke eigensheaves. Preprint available at `https://math.uchicago.edu/~Drinfel'd/langlands/QuantizationHitchin.pdf`.

[BD04] Alexander Beĭlinson and Vladimir Drinfel'd. *Chiral algebras*, volume 51 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[Gin98] Victor Ginzburg. Lectures on $\mathcal{D}$-modules, Winter 1998. Available at `https://people.mpim-bonn.mpg.de/gaitsgde/grad_2009/Ginzburg.pdf`.

[Gro68] Alexander Grothendieck. Crystals and the de Rham cohomology of schemes. In *Dix Exposés sur la Cohomologie des Schémas*, page 306–358. North-Holland, Amsterdam, 1968.

[Kap91] Mikhail M. Kapranov. On DG-modules over the de Rham complex and the vanishing cycles functor. In *Algebraic geometry (Chicago, IL, 1989)*, volume 1479 of *Lecture Notes in Math.*, pages 57–86. Springer, Berlin, 1991.

[Lur09] Jacob Lurie. Notes on crystals and algebraic $\mathcal{D}$-modules, 2009. Available at `https://people.mpim-bonn.mpg.de/gaitsgde/grad_2009/SeminarNotes/Nov17-19(Crystals).pdf`.

[Neg09] Andrei Negut. $\mathcal{D}_X$-schemes, jets and conformal blocks (the commutative case), 2009. Notes, available at `https://people.mpim-bonn.mpg.de/gaitsgde/grad_2009/SeminarNotes/Nov12(Dschemes).pdf`.

[Pos11] Leonid Positselski. Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence. *Mem. Amer. Math. Soc.*, 212(996):vi+133, 2011.

[Sai89] Morihiko Saitō. Induced $\mathcal{D}$-modules and differential complexes. *Bull. Soc. Math. France*, 117(3):361–387, 1989.

[Sim97] Carlos Simpson. The Hodge filtration on nonabelian cohomology. In *Algebraic geometry. Proceedings of the Summer Research Institute, Santa Cruz, CA, USA, July 9–29, 1995*, pages 217–281. Providence, RI: American Mathematical Society, 1997.

[Sta] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`.