Joint moments of characteristic polynomials of random unitary matrices

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Joint moments

Let $\zeta(s)$ denote the Riemann zeta function. Define Hardy's function:

$$\mathcal{Z}(t) = \pi^{-it/2} \frac{\Gamma(1/4 + it/2)}{|\Gamma(1/4 + it/2)|} \zeta(1/2 + it). \tag{1}$$

Note that $\mathcal{Z}(t)$ is real and $|\mathcal{Z}(t)| = |\zeta(1/2 + it)|$.

Joint moments

Moments and derivative moments on the critical line are of interest to number theorists. A unified approach is to study the joint moments:

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2s-2h}\,|\zeta'(1/2+it)|^{2h}dt. \tag{2}$$

(Unless otherwise stated: $s \in \mathbb{R}$, $h \in \mathbb{C}$ with $-\frac{1}{2} < \Re h < s + \frac{1}{2}$, throughout this talk).

Joint moments

A conjecture due to Hall says that:

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2s-2h} |\zeta'(1/2+it)|^{2h} dt \\ \sim C(s,h)(\log(T))^{s^2+2h}, \quad (3)$$

for an interesting, though unidentified function $C(s,h)$. For integer s, h, we can rewrite:

$$\frac{1}{T}\int_0^T |\mathcal{Z}(t)|^{2s-2h} |\mathcal{Z}'(t)|^{2h} dt \sim \tilde{C}(s,h)(\log(T))^{s^2+2h}, \quad (4)$$

where, for integer h there is a simple relation between $C(s,h)$ and $\tilde{C}(s,h)$. We are interested in (4) for $-\frac{1}{2} < \Re h < s + \frac{1}{2}$.

Values of $\tilde{C}(s,h)$

For certain values of s, h, this conjecture is solved:

(s,h)	$\tilde{C}(s,h)$	Authors
$(1,0)$	1	Hardy & Littlewood (1918)
$(2,0)$	$1/(2\pi^2)$	Ingham (1926)
$(1,1)$	$1/12$	Ingham (1926)
$(2,1)$	$1/(120\pi^2)$	Conrey (1988)
$(2,2)$	$1/(1120\pi^2)$	Conrey (1988)
$(1,1/2)$	$(e^2-5)/(4\pi)$	Conrey & Ghosh (1989)

The characteristic polynomial of a random unitary matrix.

Let $\mathbf{U}$ be a random Unitary matrix chosen according to the Haar measure $\mu_{\mathbb{U}(N)}$ on $\mathbb{U}(N)$, with eigenvalues $e^{i\theta_1}, \ldots, e^{i\theta_n}$.
Let us define

$$\mathbf{C}_{\mathbf{U}}(\theta) := \det(I - e^{-i\theta}\mathbf{U}) \tag{5}$$

and

$$\mathbf{Z}_{\mathbf{U}}(\theta) := e^{\frac{iN}{2}(\theta+\pi) - i\sum_{k=1}^{N}\frac{\theta_k}{2}}\mathbf{C}_{\mathbf{U}}(\theta) \tag{6}$$

This is real valued and $|\mathbf{Z}_{\mathbf{U}}(\theta)| = |\mathbf{C}_{\mathbf{U}}(\theta)|$.

The Keating-Snaith conjecture

There is evidence that Riemann zeros and eigenangles of random unitary matrices have similar statistical properties. An example would be the pair correlations (**Montgomery's pair correlation conjecture**),

$$\lim_{T\to\infty} \# \left\{ 0 \le \gamma, \gamma' \le T : \alpha < (\gamma - \gamma') \frac{\log(T/2\pi)}{2\pi} < \eta \right\} = \int_{\alpha}^{\eta} \left(1 - \left(\frac{\sin(\pi x)}{x} \right)^2 + \delta_0(x) \right) dx, \quad (7)$$

The Keating-Snaith conjecture

There is evidence that Riemann zeros and eigenangles of random unitary matrices have similar statistical properties. An example would be the pair correlations (**Montgomery's pair correlation conjecture**), versus the **pair correlations of eigenangles** of a random unitary matrix:

$$\lim_{N\to\infty} \mathbb{E}_{\mu_{\mathbb{U}(N)}} \left[\# \left\{ (\theta_i, \theta_j) : \alpha < \frac{N}{2\pi}(\theta_i - \theta_j) < \eta \right\} \right] = \int_\alpha^\eta \left(1 - \left(\frac{\sin(\pi x)}{x} \right)^2 + \delta_0(x) \right) dx, \quad (8)$$

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So, when we model Riemann zeros up to T by eigenangles of a random unitary matrix, we choose the scaling

$$N = \log(T/2\pi). \quad (9)$$

The Keating-Snaith conjecture

Now the idea is to model $\mathcal{Z}(t)$ by $\mathbf{Z}_{\mathbf{U}}(0)$, for the purpose of moment computations. Consider

$$\begin{aligned} \mathcal{F}(s) &= \int_{\mathbb{U}(N)} \left|\mathbf{Z}_{\mathbf{U}}(0)\right|^{2s} d\mu_{\mathbb{U}(N)} \\ &\sim \frac{G(s+1)^2}{G(2s+1)} N^{s^2}, \end{aligned} \tag{10}$$

(last line from Selberg's integral). Here G is the Barnes G-function:

$$G(z+1) = \Gamma(z)G(z). \tag{11}$$

The Keating-Snaith conjecture

Now the idea is to model $\mathcal{Z}(t)$ by $\mathbf{Z_U}(0)$, for the purpose of moment computations. Consider

$$\begin{aligned}\mathcal{F}(s) &= \int_{\mathbb{U}(N)} |\mathbf{Z_U}(0)|^{2s} \, d\mu_{\mathbb{U}(N)} \\ &\sim \frac{G(s+1)^2}{G(2s+1)} N^{s^2},\end{aligned} \tag{12}$$

(last line from Selberg's integral). We might guess that:

$$\frac{1}{T}\int_0^T |\mathcal{Z}(t)|^{2s} dt \sim \frac{G(s+1)^2}{G(2s+1)} (\log(T))^{s^2} \tag{13}$$

In fact this is not quite true, but:

The Keating-Snaith conjecture

Conjecture (Keating-Snaith).

$$\frac{1}{T}\int_0^T |\mathcal{Z}(t)|^{2s} dt \sim a(s)\frac{G(s+1)^2}{G(2s+1)}(\log(T))^{s^2}, \tag{14}$$

where the "arithmetic factor" $a(s)$ is given explictly as:

$$a(s) := \prod_p \left(1-p^{-1}\right)^{s^2} \sum_{k=0}^{\infty} p^{-k}\left(\frac{\Gamma(k+s)}{\Gamma(k+1)\Gamma(s)}\right)^2. \tag{15}$$

In other words,

$$\tilde{C}(s,0) = a(s)\frac{G(s+1)^2}{G(2s+1)}. \tag{16}$$

Hughes' conjecture

The idea now is to upgrade Keating-Snaith to joint moments. Define:

$$\mathcal{F}_N(s,h) := \int_{\mathbb{U}(N)} |\mathbf{Z}_{\mathbf{U}}(0)|^{2s-2h} \left|\mathbf{Z}'_{\mathbf{U}}(0)\right|^{2h} d\mathbf{U} \tag{17}$$

and define

$$\mathcal{F}(s,h) := \lim_{N\to\infty} \mathcal{F}_N(s,h)/N^{s^2+2h}, \tag{18}$$

so that:

$$\mathcal{F}_N(s,h) \sim \mathcal{F}(s,h) N^{s^2+2h}. \tag{19}$$

Hughes's conjecture

In the spirit of Keating and Snaith we have:

Conjecture (Hughes).

$$\frac{1}{T}\int_0^T |\mathcal{Z}(t)|^{2s-2h} \left|\mathcal{Z}'(t)\right|^{2h} dt \sim a(s)\mathcal{F}(s,h)(\log(T))^{s^2+2h}, \quad (20)$$

where $a(s)$ is as before. In other words,

$$\tilde{C}(s,h) = a(s)\mathcal{F}(s,h). \quad (21)$$

The remainder of this talk, will be about attempts to evaluate $\mathcal{F}(s,h)$.

History on $\mathcal{F}(s, h)$

Range of (s, h)	Expressions found
$(\mathbb{Z}, \mathbb{Z})$	Hughes (2006), Conrey, Rubenstein & Snaith (s=h) (2006), Dehaye (2008), Dehaye (2010)
$(\mathbb{Z}, \mathbb{Z} + 1/2)$	Winn (2012)
$(\mathbb{R}, \mathbb{Z})$	Combine Assiotis, Keating & Warren (2020) with Dehaye (2010)
$(\mathbb{Z}, \mathbb{C})$	Assiotis, Bedert, Gunes, S. (2021)
$(\mathbb{Z}, \mathbb{C})$ general β	Forrester (2021)
$(\mathbb{R}, \mathbb{Z})$ general β	Forrester (2021) Assiotis, Gunes, S. (2021)

History on $\mathcal{F}(s, h)$

There is a connection to Painlevé (that I will say more about later). Roughly, that $\mathcal{F}(s, h) \approx$ fractional derivatives at 0 of a function $\phi^{(s)}(t)$ whose log derivative solves a σ-Painlevé III$'$ equation.

Range of s	Painlevé connection established
$\mathbb{R}$	Forrester and Witte (2006)
$\mathbb{Z}$	Basor, Bleher, Buckingham, Grava, Its, Its & Keating (2018) Bailey, Bettin, Blower, Conrey, Prokhorov, Rubinstein & Snaith (2019) Assiotis, Keating & Warren (2020)
$\mathbb{R}$	Assiotis, Bedert, Gunes, S. (2021)

Understanding $\mathcal{F}(s, h)$

It was shown by Assiotis, Keating, and Warren that:

$$\mathcal{F}(s,h) = \frac{G(s+1)^2}{G(2s+1)} 2^{-2h} \mathbb{E}\left[|\mathbf{X}(s)|^{2h} \right] \quad \text{for } 0 \le \Re h < s + \frac{1}{2}, \tag{22}$$

for a particular random variable $\mathbf{X}(s)$.
$\mathbf{X}(s)$ is defined as the principal value sum over a particular determinantal point process $\mathbf{C}^{(s)}$ on $\mathbb{R}$:

$$\mathbf{X}(s) = \lim_{N\to\infty} \left[\sum_{x \in \mathbf{C}^{(s)}} x \mathbf{1}\left(|x| > \frac{1}{N^2} \right) \right]. \tag{23}$$

The random variables $\mathbf{X}(s)$

Let $\mathbf{H}_N$ be a random Hermitian matrix from the Hua-pickrell ensemble, i.e. a random matrix chosen according to the law

$$\text{const} \cdot \det\left(\left(1+\mathbf{H}^2\right)^{-s-N}\right) \times d\mathbf{H} \tag{24}$$

on $\mathbb{H}(N)$, so that the law of the eigenvalues is given by

$$\text{const}' \cdot \Delta(\mathbf{x})^2 \prod_{j=1}^{N} \left(1+x_j^2\right)^{-s-N} dx_j. \tag{25}$$

Then we have the following convergence in distribution (Qiu (2020)):

$$\frac{1}{N} \mathrm{Tr}\left(\mathbf{H}_N\right) \xrightarrow[N\to\infty]{d} \mathbf{X}(s). \tag{26}$$

The random variables $\mathbf{X}(s)$

Then we have the following convergence in distribution:

$$\frac{1}{N}\mathrm{Tr}\left(\mathbf{H}_N\right) \xrightarrow[N\to\infty]{d} \mathbf{X}(s). \tag{27}$$

and hence the convergence of characteristic functions:

$$\phi_N^{(s)}(t) := \mathbb{E}_N^{(s)}\left[e^{\frac{it}{2N}\mathrm{Tr}(\mathbf{H}_N)}\right] \xrightarrow{N\to\infty} \mathbb{E}\left[e^{\frac{it}{2}\mathbf{X}(s)}\right] =: \phi^{(s)}(t). \tag{28}$$

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Idea

By taking the $N \to \infty$ limit of a formula for $\phi_N^{(s)}(t)$ we will get a formula for $\phi^{(s)}(t)$. The density of $\mathbf{X}(s)$ can be recovered by Fourier inversion. Integrating against x^h, we get the moments and hence $\mathcal{F}(s,h)$.

Theorem 1: Expressions for $\mathcal{F}(s,h)$.

Theorem (Assiotis, Bedert, Gunes, S. (2021), Forrester (2021).)

For $s \in \mathbb{N} \cup \{0\}$, the density $\rho^{(s)}(x)$ of $\mathbf{X}(s)$ is given by:

$$\rho^{(s)}(x) = \frac{1}{2\pi} \Re \left\{ \sum_{\substack{\lambda \in \mathbb{Y} \\ \ell(\lambda) \leq s}} \frac{[s]_\lambda}{[2s]_\lambda h_\lambda^2} \cdot \left(\frac{2}{1 - ix} \right)^{|\lambda|+1} \right\}. \tag{29}$$

For $s \in \mathbb{N} \cup \{0\}, \Re(h) \in (-\frac{1}{2}, s + \frac{1}{2})$, we have:

$$\mathcal{F}(s,h) = \frac{1}{2^{2h} \cos(\pi h)} \frac{G(s+1)^2}{G(2s+1)} \sum_{\substack{\lambda \in \mathbb{Y} \\ \ell(\lambda) \leq s}} \frac{[s]_\lambda 2^{|\lambda|} (-2h)_{|\lambda|}}{[2s]_\lambda h_\lambda^2}. \tag{30}$$

Explanation of notation

- $\lambda =$ integer partition $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$, equiv. Young diagram; boxes indexed by coördinates (i, j),
- $\ell(\lambda) = n$ the length,
- $|\lambda| = \lambda_1 + \cdots + \lambda_n$,
- $(a)_k := \frac{\Gamma(a+k)}{\Gamma(a)}$ is the Pochhammer symbol,
- $[x]_\lambda := \prod_{i=1}^{\ell(\lambda)} (x - i + 1)_{\lambda_i}$ is the generalised Pochammer symbol associated to λ,
- h_λ is the hook-length of the partition.

Real s and integer h.

Remark

From these results (also those of Dehaye), it follows that for s, h integer, $s \geq h - \frac{1}{2}$, $\mathcal{F}(s,h)$ (ignoring prefactors) is a rational function in s.
Assiotis-Keating-Warren showed that for integer h and real $s \geq h - \frac{1}{2}$, $\mathcal{F}(s,h)$ (removing prefactors) is a rational function in s.
So can recover the expression, valid for $h \in \mathbb{N}$ and $s \in (h - \frac{1}{2}, \infty)$:

$$\mathcal{F}(s,h) = \frac{(-1)^h}{2^{2h}} \frac{G(s+1)^2}{G(2s+1)} \sum_{\substack{\lambda \in \mathbb{Y} \\ |\lambda| \leq 2h}} \frac{[s]_\lambda 2^{|\lambda|} (-2h)_{|\lambda|}}{[2s]_\lambda h_\lambda^2}. \tag{31}$$

Special cases of Theorem 1

Corollary

The densities of $\mathbf{X}(0), \mathbf{X}(1)$, and $\mathbf{X}(2)$ on $\mathbb{R}$ are given by:

$$\rho_N^{(0)}(x) = \rho^{(0)}(x) = \frac{1}{\pi(1+x^2)}$$

$$\rho^{(1)}(x) = \frac{1}{2\pi}\left(-1 + e^{\frac{2}{1+x^2}} \cos\left(\frac{2x}{1+x^2}\right)\right)$$

$$\rho^{(2)}(x) = \frac{1}{\pi}\Re\left\{\frac{1}{1-ix}\, {}_2F_2\left[\begin{matrix}\frac{5}{2}, 1 \\ 5, 4\end{matrix} \,\middle|\, \frac{8}{1-ix}\right]\right\}$$

Here $\rho_N^{(s)}(x)$ denotes the finite-N density.

Special cases of Theorem 1

Using these (or from Theorem 1 directly) we can get the moments:

Corollary

$$\frac{\mathcal{F}_N(0,h)}{N^{2h}} = \mathcal{F}(0,h) = 2^{-2h}\frac{1}{\cos(\pi h)} \qquad -\frac{1}{2} < \Re(h) < \frac{1}{2}$$

$$\mathcal{F}(1,h) = 2^{-2h}\frac{1}{\cos(\pi h)}\,{}_1F_1\left[\begin{matrix}-2h\\ 2\end{matrix}\;\middle|\;2\right] \qquad -\frac{1}{2} < \Re(h) < \frac{3}{2}$$

$$\mathcal{F}(2,h) = 2^{-2h}\frac{1}{12\cos(\pi h)}\,{}_2F_2\left[\begin{matrix}\frac{5}{2},-2h\\ 5,4\end{matrix}\;\middle|\;8\right] \qquad -\frac{1}{2} < \Re(h) < \frac{5}{2}$$

Special cases of Theorem 1

By using L'Hôpital's rule, we recover the expressions at half-integer h, originally found by Winn (2012):

Corollary

$$\mathcal{F}\left(1,\frac{1}{2}\right)=\frac{e^2-5}{4\pi},$$
$$\mathcal{F}\left(2,\frac{1}{2}\right)=\frac{7}{180\pi}\left(\frac{15}{7}-{}_3F_3\left[\left.\begin{matrix}\frac{9}{2},1,1\\3,6,7\end{matrix}\right|8\right]\right), \tag{32}$$
$$\mathcal{F}\left(2,\frac{3}{2}\right)=\frac{11}{3360\pi}\left(-\frac{28}{33}+{}_3F_3\left[\left.\begin{matrix}\frac{13}{2},1,1\\5,8,9\end{matrix}\right|8\right]\right).$$

Winn's strategy (also Forrester (2021)).

For a generalisation of the following arguments to arbitary $\beta > 0$: See Peter J. Forrester, Joint moments of a characteristic polynomial and its derivative for the circular β-ensemble

Winn's strategy (also Forrester (2021)).

Step 1. For $t > 0, s \in \left(-\frac{1}{2}, \infty\right)$, the following equality of integrals was shown by Winn (2012):

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{N} \frac{e^{itx_j}}{(1 + x_j{}^2)^{s+N}} \Delta(\mathbf{x})^2 d\mathbf{x}$$

$$= \frac{\pi^N}{2^{(N+2s-1)N}} \prod_{j=0}^{N-1} \frac{1}{\Gamma\left(s + 1 + j\right)^2} \cdot e^{-Nt}$$

$$\times \int_0^{\infty} \cdots \int_0^{\infty} \prod_{j=1}^{N} \left(y_j + 2t\right)^s y_j{}^s e^{-y_j} \Delta(\mathbf{y})^2 d\mathbf{y}. \quad (33)$$

Winn's strategy (also Forrester (2021)).

Step 2. Taking limits of a "beta integral" formula of Forrester and Keating (2004), Winn shows, for $s \in \mathbb{N} \cup \{0\}, t > 0$, that:

$$\int_0^\infty \prod_{j=1}^{N} (y_j + 2t)^s \, {y_j}^s e^{-y_j} \Delta(\mathbf{y})^2 d\mathbf{y}$$

$$= \prod_{j=1}^{s} \Gamma(j+1)\Gamma(j+s)(2t)^{sN}$$

$$\times {}_2F_0\left[\begin{matrix} -s, N+s \\ . \end{matrix} \;\middle|\; \underbrace{\frac{-1}{2t}, \ldots, \frac{-1}{2t}}_{N \ \ times} \right]. \quad (34)$$

Winn's strategy (also Forrester (2021)).

By the reflection formula relating ${}_2F_0$ to ${}_1F_1$, we get the following evaluation for $s \in \mathbb{N} \cup \{0\}, t \in \mathbb{R}$:

$$\begin{aligned} \phi_N^{(s)}(t) &= e^{-|t|}{}_1F_1\left[\begin{array}{c|c} -s & \underbrace{\frac{2|t|}{N}, \dots, \frac{2|t|}{N}}_{N \ \ times} \\ -2s & \end{array}\right] \\ &= e^{-|t|} \sum_{\substack{\lambda \in \mathbb{Y} \\ \lambda_1 \le s}} \frac{[-s]_\lambda [N]_\lambda}{[-2s]_\lambda h_\lambda^2} \left(\frac{2|t|}{N}\right)^{|\lambda|} \end{aligned} \tag{35}$$

where ${}_1F_1$ denotes the confluent hypergeometric function of matrix argument.

Finishing off Theorem 1 (also Forrester (2021)).

Step 3. Take the limit as $N \to \infty$ termwise:

$$\phi^{(s)}(t) = e^{-|t|} \sum_{\substack{\lambda \in \mathbb{Y} \\ \ell(\lambda) \le s}} \frac{[s]_\lambda 2^{|\lambda|}}{[2s]_\lambda h_\lambda^2} |t|^{|\lambda|}. \tag{36}$$

Now Fourier inversion yields the density $\rho^{(s)}(x)$ of $\mathbf{X}(s)$:

$$\rho^{(s)}(x) = \frac{1}{2\pi} \Re \left\{ \sum_{\substack{\lambda \in \mathbb{Y} \\ \ell(\lambda) \le s}} \frac{[s]_\lambda}{[2s]_\lambda h_\lambda^2} \cdot \left(\frac{2}{1 - ix} \right)^{|\lambda|+1} \right\}. \tag{37}$$

Finishing off Theorem 1 (also Forrester (2021)).

Finally, integrating against $|x|^{2h}$ yields the moments:

$$\mathbb{E}\left[|\mathbf{X}(s)|^{2h}\right] = \frac{1}{\cos(\pi h)} \sum_{\substack{\lambda\in\mathbb{Y} \\ \ell(\lambda)\leq s}} \frac{[s]_\lambda 2^{|\lambda|}(-2h)_{|\lambda|}}{[2s]_\lambda h_\lambda^2}, \tag{38}$$

therefore,

$$\mathcal{F}(s,h) = \frac{1}{2^{2h}\cos(\pi h)} \frac{G(s+1)^2}{G(2s+1)} \sum_{\substack{\lambda\in\mathbb{Y} \\ \ell(\lambda)\leq s}} \frac{[s]_\lambda 2^{|\lambda|}(-2h)_{|\lambda|}}{[2s]_\lambda h_\lambda^2}. \tag{39}$$

Alternative approach for Theorem 1

Our starting point is following evaluation, also given by Winn, for $s \in \mathbb{N}$:

$$\phi_N^{(s)}(t) = (-1)^{s(s-1)/2} \prod_{j=0}^{N-1} \frac{\Gamma(s+N-j)^2}{j!\Gamma(2s+N-j)} e^{-|t|/2} \times \det\left[L_{N+s-1-i-j}^{2s-1}\left(-\frac{|t|}{N}\right)\right]_{i,j=0}^{s-1}, \quad (40)$$

where $L_n^{(\alpha)}(x)$ denotes the Laguerre polynomial of order n and parameter α.

Alternative approach for Theorem 1

Convergence of the logarithmic derivative of this determinant was proven rigorously using Riemann-Hilbert problem methods in Basor, Bleher, Buckingham, Grava, Its, Its, Keating (2018). In our language:

$$\frac{d}{dt}\log\phi_N^{(s)}(t) \xrightarrow{N\to\infty} \frac{d}{dt}\log\left(\frac{\det\left[I_{j+k+1}\left(2\sqrt{|t|}\right)\right]_{j,k=0}^{s-1}}{e^{|t|/2}|t|^{s^2/2}}\right).$$

We show that $\phi_N^{(s)}$ and its derivatives converge to those of $\phi^{(s)}$, hence, evaluating at 0 we get

$$\phi^{(s)}(t) = (-1)^{s(s-1)/2}\frac{G(2s+1)}{G(s+1)^2}\times\frac{\det\left[I_{j+k+1}\left(2\sqrt{|t|}\right)\right]_{j,k=0}^{s-1}}{e^{|t|/2}|t|^{s^2/2}}.$$

Theorem 2: Painlevé equation

Theorem

Let $s > -\frac{1}{2}$ *and define* $\tau^{(s)}(t) := t\frac{d}{dt}\log(\phi^{(s)}(t))$. *Then* $\tau^{(s)}(t)$ *is* C^{ω} *on* $\mathbb{R}^*$ *and is a solution to a special case of the* σ*-Painlevé III' equation with two parameters for* $t \in \mathbb{R}^*$*:*

$$\left(t\frac{d^2\tau^{(s)}}{dt^2}\right)^2 = -4t\left(\frac{d\tau^{(s)}}{dt}\right)^3 + \left(4s^2 + 4\tau^{(s)}\right)\left(\frac{d\tau^{(s)}}{dt}\right)^2 + t\frac{d\tau^{(s)}}{dt} - \tau^{(s)}. \quad (41)$$

$$\text{with BCs: } \begin{cases} \tau^{(s)}(0) = 0, & \text{for } s > 0, \\ \frac{d}{dt}\tau^{(s)}(t)\Big|_{t=0} = 0, & \text{for } s > \frac{1}{2}. \end{cases} \quad (42)$$

Implications of Theorem 2.

In short: Painlevé equations define new functions.

Let S be a set of meromorphic functions on a domain $D \subseteq \mathbb{C}$.

Permissible operations on S are:

- Taking algebraic roots, i.e. $f^n + a_1 f^{n-1} + \cdots + a_n = 0$, $a_i \in S$.
- Taking primitives of functions in S.
- Taking solutions of linear differential equations with coefficients in S.
- Functions of the form $\phi \circ \pi(f_1, \ldots, f_n)$, where $\pi : \mathbb{C}^n \to \mathbb{C}^n/\Lambda$ is the projection, and ϕ is holomorphic on $\mathbb{C}^n/\Lambda$.

Implications of Theorem 2

Definition

A meromorphic function f on D is called classical if $\exists$ a tower $\mathbb{C}(t) = K_0 \subset \cdots \subset K_m \subset \mathcal{M}_D$ of differential fields such that:

- $K_j = K_{j-1}(g_j, g_j', \dots)$ for some g_j obtained from K_{j-1} by permissible operations;
- $f \in K_m$.

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According to results of Umemura and Watanabe (1998), the only classical solutions to our Painlevé that aren't algebraic, occur when $s \in \mathbb{N}$.

By our "Alternative approach" to Theorem 1, we saw that our solutions are classical.

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By our "Alternative approach" to Theorem 1, we saw that our solutions are classical.

Do we need a new idea to calculate $\mathcal{F}(s, h)$, when neither s, h are integral?