# $p$-adic Hodge tate talk

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**Abstract**

Let me know if there are mistakes and typos

## Contents



## 1 Recap

Let $K$ be a $p$-adic field. Recall the 1-d Tate module $\mathbb{Z}_p(1)$ with a choice of generator $t \in \mathbb{Z}_p(1)$. It is a $G_K$-module, with action given by:

$$g(t) = \chi(g)t, \tag{1}$$

and $\mathbb{Z}_p(i)$ $(i \in \mathbb{Z})$ is the free $\mathbb{Z}_p$-module with generator $t^i$ where $G_K$ acts by $\chi^i$. Recall also, if $M \in G_K - \text{mod}$ we define its Tate twist $M(i) = M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i)$.

Set $\mathbb{C}_K := \widehat{\overline{K}}$, which is a $G_K$-module since $G_K$ can be identified with the group of isometric isomorphisms of $\mathbb{C}_K$. With this in mind, define the *Hodge-Tate period ring* $B_{HT}$:

$$B_{HT} = \bigoplus_{q \in \mathbb{Z}} \mathbb{C}_K(q) \simeq \mathbb{C}_K[t, t^{-1}]. \tag{2}$$

It is a graded ring, the multiplication comes from the maps $\mathbb{C}_K(q) \otimes \mathbb{C}_K(q') \to \mathbb{C}_K(q+q')$. The second isomorphism is the map from $c \otimes t^i \to ct^i$. You can see the $G_K$-action respects the grading. That $(B_{HT})^{G_K} \simeq K$ follows from:

**Theorem 1.1** (Tate-Sen)**.** *For $i = 0, 1$ and any continuous character $\eta : G_K \to \mathbb{Z}_p^\times$, we have:*

$$H^i(G_K, \mathbb{C}_K(\eta)) \cong \begin{cases} 0 \text{ if } \eta(I_K) \text{ infinite,} \\ K \text{ if } \eta(I_K) \text{ finite.} \end{cases} \tag{3}$$

*Of particular use is:*

$$H^i(G_K, \mathbb{C}_K(n)) \cong \begin{cases} 0 & \text{if } n \neq 0 \\ K & \text{if } n = 0. \end{cases} \tag{4}$$

## 2 The equivalence of categories

We define the category:

$$\mathsf{Rep}_{\mathbb{C}_K}(G_K) = \left\{ \begin{array}{c} \text{f.d. } \mathbb{C}_K\text{-vector spaces } W \text{ equipped with} \\ \text{a continuous } \mathbb{C}_K\text{-semilinear } G_K\text{-action.} \end{array} \right\} \tag{5}$$

It is an abelian category endowed with tensors, direct sums, and duality satisfying all the usual properties. Semilinear means $g(cw) = g(c)g(w)$, for $c \in \mathbb{C}_K$ and $w \in W$. We define:

$$W\{q\} := W(q)^{G_K}, \tag{6}$$

this is a $K$-vector space. By left exactness of $(-)^{G_K}$ and the flat extension of scalars $K(-q) \otimes_K -$, we get an injection ($K$-linear, $G_K$-equivariant, where it's acting diagonally):

$$K(-q) \otimes_K W\{q\} \hookrightarrow K(-q) \otimes W(q) \simeq W, \tag{7}$$

the last isomorphism is from multiplication. Extending further to $\mathbb{C}_K$, we get maps $\mathbb{C}_K(-q) \otimes_K W\{q\} \hookrightarrow W$. Lastly, summing over all $q$, we get a map:

$$\xi_W : \bigoplus_q \mathbb{C}_K(-q) \otimes_K W\{q\} \to W. \tag{8}$$

The important lemma is:

**Lemma 2.1** (Serre-Tate)**.** *$\xi_W$ is injective.*

Therefore, $\sum_q \dim_K W\{q\} \leq \dim_{\mathbb{C}_K} W$, and you see that equality here is the same as $\xi_W$ being an isomorphism.

**Definition 2.2.** *$W \in \mathsf{Rep}_{\mathbb{C}_K}(G_K)$ is Hodge-Tate if $\xi_W$ is an isomorphism. $\mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K)$ is the full subcategory of Hodge-Tate objects.*

In which case, we define the Hodge-Tate weights $h_q = \dim_K W\{q\}$ for all $q$ where this isn't 0.

[**Aside**: choosing a basis in each $W\{q\}$ gives a (non-canonical) isomorphism

$$W \cong \bigoplus_q \mathbb{C}_K(-q)^{h_q}. \tag{9}$$

The Tate-Sen theorem then shows that this can be taken as a definition of Hodge-Tate. By this description, it's easy to see that $\mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K)$ is closed under tensors and direct sums. The dual has the negated weights.]

As usual, we are going to translate $\mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K)$ into "semilinear algebraic data". For $W \in \mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K)$ consider

$$\underline{D}(W) = (B_{HT} \otimes W)^{G_K} = \bigoplus_q W\{q\}. \tag{10}$$

This defines a functor, and the description on the left, together with Lemma 2.1, shows us what the target category is:

$$\underline{D} : \mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K) \to \mathsf{Gr}_{K,f} := \left\{ \begin{array}{c} \text{f.d. } \mathbb{Z}\text{-graded } K\text{-vector spaces } D, \\ \text{morphisms are grading preserving linear maps.} \end{array} \right\} \tag{11}$$

We can go back in the reverse direction. Let $D \in \mathsf{Gr}_{K,f}$. Then $B_{HT} \otimes_K D$ is a graded $\mathbb{C}_K$-vector space[1], and we set:

$$\underline{V}(D) = \mathrm{gr}^0(B_{HT} \otimes_K D) = \bigoplus_q \mathbb{C}_K(-q) \otimes_K D_q, \tag{12}$$

which gives an exact functor $\underline{V} : \mathsf{Gr}_{K,f} \to \mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K)$. Now, we consider $\underline{V}(\underline{D}(-))$. Let $W \in \mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K)$, and consider first the following composite, $\gamma_W$:

$$\begin{aligned} \gamma_W : B_{HT} \otimes_K \underline{D}(W) \hookrightarrow B_{HT} \otimes_K (B_{HT} \otimes_{\mathbb{C}_K} W) &\to B_{HT} \otimes_{\mathbb{C}_K} W \\ a \otimes b \otimes w &\mapsto ab \otimes w. \end{aligned} \tag{13}$$

It is $G_K$-equivariant, and grading preserving. Now consider this map in degree 0. It takes:

$$\underline{V}(\underline{D}(W)) = \mathrm{gr}^0(B_{HT} \otimes_K D) = \bigoplus_q \mathbb{C}_K(-q) \otimes W\{q\} \to W, \tag{14}$$

exactly as $\xi_W$. Therefore, by Lemma 2.1 it is an isomorphism. Even more: you can see that in degree $n$, $\gamma_W$ is the $\mathbb{Z}_p(n)$-twist of $\xi_W$, and so $\gamma_W$ is an isomorphism.

Next, consider $\underline{D}(\underline{V}(D))$. Since $\underline{V}(D)$ is Hodge-Tate, we get an isomorphism ($G_K$-equivariant, grading preserving):

$$\gamma_{\underline{V}(D)} : B_{HT} \otimes_K \underline{D}(\underline{V}(D)) \simeq B_{HT} \otimes_{\mathbb{C}_K} \underline{V}(D). \tag{15}$$

Now pass to $G_K$-invariants. We get the chain of equalities:

$$\begin{aligned} \underline{D}(\underline{V}(D)) &\simeq \bigoplus_r (\underline{V}(D) \otimes_{\mathbb{C}_K} \mathbb{C}_K(r))^{G_K} \\ &= \bigoplus_r (\bigoplus_q \mathbb{C}_K(r-q) \otimes_K D_q)^{G_K} \\ &= \bigoplus_r D_r = D, \end{aligned} \tag{16}$$

where we used the Tate-Sen theorem going from the second to the third line. Therefore:

**Theorem 2.3.** *The functors $\underline{D}$ and $\underline{V}$ are quasi-inverses, setting up an equivalence of categories:*

$$\underline{D} : \mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K) \rightleftarrows \mathsf{Gr}_{K,f} : \underline{V}. \tag{17}$$

## 2.1 The category $\mathsf{Rep}^{HT}_{\mathbb{Q}_p}(G_K)$.

Recall that $\mathsf{Rep}_{\mathbb{Q}_p}(G_K)$ is the category of continuous representations of $G_K$ into f.d. $\mathbb{Q}_p$ vector spaces. (The formalism of admissible representations is directly applicable in this case but not directly for $\mathbb{C}_K$, because of the semilinearity).

**Definition 2.4.** *$V \in \mathsf{Rep}_{\mathbb{Q}_p}(G_K)$ is called Hodge-Tate if $\mathbb{C}_K \otimes_{\mathbb{Q}_p} V \in \mathsf{Rep}_{\mathbb{C}_K}(G_K)$ is Hodge-Tate. The full subcategory is denoted $\mathsf{Rep}^{HT}_{\mathbb{Q}_p}(G_K)$.*

[1] Set $\mathrm{gr}^n(B_{HT} \otimes_K D) = \oplus_q \mathbb{C}_K(n-q) \otimes_K D_q$.

We define $D_{HT} : \mathsf{Rep}^{HT}_{\mathbb{Q}_p} \to \mathsf{Gr}_{K,f}$ by:

$$D_{HT}(V) = \underline{D}(\mathbb{C}_K \otimes_{\mathbb{Q}_p} V) = (B_{HT} \otimes_{\mathbb{Q}_p} V)^{G_K}. \tag{18}$$

The functor $D_{HT}$ is faithful, but it is not full. To see this, by the Tate-Sen theorem (see talk 4), it follows that for any finitely ramified character $\eta$, $D_{HT}(\mathbb{Q}_p) \cong D_{HT}(\mathbb{Q}_p(\eta))$ but $\mathbb{Q}_p$, $\mathbb{Q}_p(\eta)$ admit no maps in $\mathsf{Rep}_{\mathbb{Q}_p}(G_K)$.

## 2.2 Properties of $D$ and $V$

I would say to probably read Brinon and Conrad if you are interested in the details of these first two.

**Proposition 2.5** (Exactness). *$\mathsf{Rep}_{\mathbb{C}_K}(G_K)$ (and so $\mathsf{Rep}_{\mathbb{Q}_p}(G_K)$) are stable under subobjects and quotients, and $\underline{D}$ (resp. $D_{HT}$) is exact on $\mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K)$ (resp. $\mathsf{Rep}^{HT}_{\mathbb{Q}_p}(G_K)$).*

**Proposition 2.6** (Compatibility with tensors and duals). *For $W$, $W' \in \mathsf{Rep}^{HT}_{\mathbb{C}_K}$, there are natural isomorphisms $\underline{D}(W \otimes W') \cong \underline{D}(W) \otimes \underline{D}(W')$ and a natural isomorphism $\underline{D}(W^\vee) \cong \underline{D}(W)^\vee$. Pretty much the same holds for $D_{HT}$.*

The above has all implicitly depended on the base field $K$. In the next proposition we make this explicit with the notation $\underline{D} = \underline{D}_K : \mathsf{Rep}_{\mathbb{C}_K}(G_K) \to \mathsf{Gr}_{K,f}$. Let $K'/K$ be finite and $\widehat{K^{un}}$ be as usual, all contained in a fixed $\overline{K} \subset \mathbb{C}_K$.

Let $W \in \mathsf{Rep}_{\mathbb{C}_K}(G_K)$. Because $G_{K'} \subset G_K$, we get a natural map $K' \otimes_K \underline{D}_K(W) \to \underline{D}_{K'}(W)$ in $\mathsf{Gr}_{K',f}$, (and the same with $\widehat{K^{un}}$). Recall $G_{\widehat{K^{un}}} = I_K$. The below says "Hodge-Tate" is the same if we pass to a finite extension, or restrict to the inertia.

**Proposition 2.7** (Scalar extension). *The just described maps in $\mathsf{Gr}_{K',f}$, $\mathsf{Gr}_{\widehat{K^{un}},f}$, are isomorphisms.*

As a warning note that $\mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K)$ is not closed under extensions. An elementary proof is available in Brinon and Conrad but it might be best explained using Sen theory.

## 2.3 Why is it called $p$-adic Hodge theory?

Faltings proved:

**Theorem 2.8.** *If $X$ is a smooth proper scheme over $K$, then the étale cohomology $H^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p) \in \mathsf{Rep}^{HT}_{\mathbb{Q}_p}(G_K)$, and $D_{HT}(H^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p)) \cong \bigoplus_q H^{i-q}(X, \Omega^q_{X/K})$, the Hodge cohomology.*

This is a $p-$adic analogue of the comparison between de Rham and singular cohomology for a smooth manifold (where the isomorphism comes from integration over cycles, Stokes's theorem).

# 3 Sen theory

The main idea of Sen theory is to differentiate the Galois action to get an operator called the Sen operator $\phi$, and then seen how this controls the decomposition. It appears to be independent of the period ring formalism. We will see that being Hodge-Tate is the same as $\phi$ acting semisimply with integer eigenvalues.

## 3.1 Setup

We begin with the following simple result:

**Proposition 3.1.** *Let $G$ be a top. group and let $B$ be a top. ring with $G \curvearrowright B$ continuously. There is a natural bijection:*

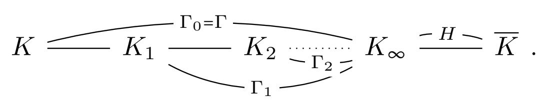
$$H^1_{\text{cont}}(G, GL_d(B)) \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Isoclasses of free continuous } B\textit{-semilinear} \\ G\textit{-representations of rank } d. \end{array} \right\} \tag{19}$$

*Proof.* Let $V$ be such a representation, and let $\alpha(g)$ be the matrix of $g$ with respect to some basis. What I mean is that

$$g(e_i) = \sum_j a_{ij}(g) e_j, \text{ for some } a_{ij}(g) \in B, \tag{20}$$

and $\alpha(g) = (a_{ij}(g))$. Then $\alpha(gh) = \alpha(g) g(\alpha(h))$ (cocycle condition). If $\alpha'$ is the matrix wrt. some other basis, and $P$ is the change of basis matrix, then $P\alpha'(g) = \alpha(g) g(P)$ (coboundary condition). Lastly, any cocycle defines a representation into $B^d$. $\square$

First, notation. $K_\infty/K$ is a ramified $\mathbb{Z}_p$-extension living inside $\overline{K}$, $H := G_{K_\infty}$, $\Gamma := \mathrm{Gal}(K_\infty/K)$, with topological generator $\gamma$, $\chi : \Gamma \to \mathbb{Z}_p^\times$ is multiplicative character, $\Gamma_m := \Gamma^{p^m}$ with topological generator $\gamma_m := \gamma^{p^m}$, and $K_m := K_\infty^{\Gamma_m}$. The most important example of this would be a cyclotomic extension. Here is a picture:



(21)

Because of Proposition 3.1 we will start looking at various homology groups. Firstly, a "strong version" of Hilbert's theorem 90:

**Proposition 3.2.** $H^1_{\text{cont}}(H, GL_d(\mathbb{C}_K)) = 1$. *Therefore, by an "inflation-restriction" exact sequence (see Weibel 6.7.3), we get an iso:*

$$j : H^1_{\text{cont}}(\Gamma, GL_d(\widehat{K_\infty})) \simeq H^1_{\text{cont}}(G_K, GL_d(\mathbb{C}_K)) \tag{22}$$

We also have a "decompletion" result:

**Proposition 3.3.** *The natural map*

$$\iota : H^1_{\text{cont}}(\Gamma, GL_d(K_\infty)) \to H^1_{\text{cont}}(\Gamma, GL_d(\widehat{K_\infty})) \tag{23}$$

*is an isomorphism, and any cocycle in $H^1_{\text{cont}}(\Gamma, GL_d(K_\infty))$ is cohomologous to a cocycle with values in $GL_d(K_r)$, if we take $r$ large enough.*

At the level of isoclasses of semilinear reps, this amounts to the following. Let $W$ be a $d$–dimensional $\mathbb{C}_K$-semilinear rep of $H$, and set $\widehat{W}_\infty = W^H$. By Proposition 3.2, $W \simeq \widehat{W}_\infty \otimes_{\widehat{K_\infty}} \mathbb{C}_K$, and by Proposition 3.3, we can chase the isoclass $[\widehat{W}_\infty]$ back, to a representation $W_r$ defined over some $K_r$ such that:

$$W_r \otimes_{K_r} \widehat{K}_\infty \simeq \widehat{W}_\infty. \tag{24}$$

(Think of it like, the maps between the $H^1_{\text{cont}}$'s are extension of scalars, (when we view them as the isoclasses), and Galois descent is what undoes this). Now set

$$W_\infty = \{K\text{-finite vectors } w \in \widehat{W_\infty}\}, \tag{25}$$

where $K$-finite means that $\dim_K K\text{-span}(\Gamma w) < \infty$. This is a $\Gamma$-stable $K_\infty$-vector space, containing $W_r$, and it is $d$–dimensional by a short argument using (24). Therefore, we have four $d$-dimensional semilinear reps over $K_r, K_\infty, \widehat{K_\infty}, \mathbb{C}_K$, respectively:

$$W_r \to W_\infty \to \widehat{W_\infty} \to W \tag{26}$$

where each is isomorphic to the next after extending scalars and inflating to the larger Galois group.

## 3.2 The Sen operator $\phi$

Now, fix a $K_r$-basis $\{e_1, \dots, e_d\}$ of $W_r$. It will also be a $K_\infty$-basis of $W_\infty$ and a $\mathbb{C}_K$-basis of $W$. Let $\rho : \Gamma_r \to GL_d(K_r)$ be the matrix wrt this basis.

**Definition 3.4.** *The Sen operator $\phi$ associated to $W$ is the linear endomorphism of $W_r$ whose matrix wrt this basis is given by:*

$$\Phi = \log(\rho(\gamma_r)) / \log(\chi(\gamma_r)), \tag{27}$$

*and its linear extensions to $W_\infty$, and $W$.*

I am glossing over the fact that you can do these log's (because $\nu(\rho(\gamma_r) - 1) > c + d$, for $c, d \in \mathbb{Z}$ which come from Tate's normalised traces).

The main theorem is the following alternative characterisation:

**Theorem 3.5.** *The Sen operator $\phi$ is the unique $K_\infty$-linear endomorphism of $W_\infty$ with the following property.*

*For all $w \in W_\infty$, there is an open subgroup $\Gamma_\omega$ of $\Gamma$ such that:*

$$\sigma(w) = \exp(\phi \log \chi(\sigma)) w \tag{28}$$

*for all $\sigma \in \Gamma$.*

The expression $\exp(\phi \log \chi(\sigma))$ is a $K_\infty$-linear endomorphism.

*Proof.* Write $w = \lambda_1 e_1 + \cdots + \lambda_d e_d$. There are $r_1, \dots, r_d$ such that $\lambda_i \in K_{r_i} = K_\infty^{\Gamma_{r_i}}$ (this is where we use the $K$-finiteness). Set $\Gamma_w = \Gamma_r \cap \Gamma_{r_1} \cap \cdots \cap \Gamma_{r_n}$. Every $\sigma \in \Gamma_w$ takes the form $\sigma = \gamma_r^a$ for some $a \in \mathbb{Z}_p$. Because $\rho(\gamma_r)$ takes values in $K_r$, we have that $\rho(\sigma) = \rho(\gamma_r)^a$. Now, as matrices with entries in $K_r$, we have:

$$\exp(\Phi \log \chi(\sigma)) = \exp(a \log \rho(\gamma_r)) = \rho(\gamma_r)^a = \rho(\sigma), \tag{29}$$

and because all the $\lambda_i$ are fixed by $\sigma$, it follows that $\exp(\phi \log \chi(\sigma)) w = \sigma(w)$, for all $\sigma \in \Gamma_w$. I am omitting uniqueness but it is not hard, maybe you can see it already. ☐

We may use the notation $\phi_W$ to denote dependence on $W$. In that case $\phi_{W_1 \oplus W_2} = \phi_{W_1} \oplus \phi_{W_2}$, $\phi_{W_1 \otimes W_2} = \phi_{W_1} \otimes 1 + 1 \otimes \phi_{W_2}$, and $\phi_{\text{Hom}(W_1, W_2)} = (\phi_{W_1})^* - (\phi_{W_2})_*$. Now, it follows from the formula (28) that for $w \in W_\infty$:

$$\phi(w) = \frac{1}{\log \chi(\gamma)} \frac{d}{dx}\Big|_{x=0} (\gamma^x w) = \frac{1}{\log \chi(\gamma)} \lim_{n \to \infty} \frac{\gamma^{p^n}(w) - w}{p^n}. \tag{30}$$

It follows from this expression that $\phi$ is $\Gamma$-linear on $W_\infty$ and $G_K$-linear on $W$. Also, if $w = t \in \mathbb{C}_K(q)$, then $w$ is $K$-finite, and we calculate:

$$\phi(w) = \frac{1}{\log(\chi(\gamma))} \left.\frac{d}{dx}\right|_{x=0} (\chi(\gamma)^{qx} w) = qw. \tag{31}$$

So $\phi$ is just multiplication by $q$ on $\mathbb{C}_K(q)$. Thus we see that $\phi$ acts semisimply with integer coefficients, if $W$ is Hodge-Tate. We aim to prove the converse.

**Theorem 3.6.** $\ker\phi = W^{G_K} \otimes_K \mathbb{C}_K$.

*Proof.* The formula (30) shows that $G$-invariants belong to the kernel. Because $\phi$ is $G_K$-linear, $\ker\phi$ is $G_K$-stable. So consider $(\ker\phi)_\infty$ as before: we have $(\ker\phi)_\infty \otimes_{K^\infty} \mathbb{C}_K = \ker\phi$, and the Sen operator (which is 0), just comes from the one on $(\ker\phi)_\infty$ extended linearly. But by looking at formula (30) for one direction, and Theorem 3.5 for the other, we can see that $\phi(w) = 0$ is equivalent to $\Gamma w$ being finite, equivalently, the $\Gamma$-action is continous for the discrete topology on $(\ker\phi)_\infty$, equivalently, the $\Gamma$-action factors through an open subgroup $\Gamma_r$ of $\Gamma$. Therefore, combining Hilbert's theorem 90 (that $H^1(\Gamma/\Gamma_r, GL_n(K_\infty)) = 1$) with Proposition 3.1 shows that $(\ker\phi)_\infty$ has a basis of $G_K$-invariants. ☐

Now, using this, for $q \in \mathbb{Z}$ we can naturally identify $\ker(\phi + qI) = W(q)^{G_K} \otimes_K \mathbb{C}_K = W\{q\}$, whence it follows that:

**Theorem 3.7.** *$W$ is Hodge-Tate iff $\phi$ acts semisimply with integer eigenvalues.*

By applying Theorem 3.6 to the representation $\mathrm{Hom}_{\mathbb{C}_K}(W_1, W_2)$ one can deduce:

**Proposition 3.8.** *$W_1, W_2 \in \mathsf{Rep}_{\mathbb{C}_K}(G_K)$ are isomorphic iff $\phi_{W_1}$ is similar to $\phi_{W_2}$.*

**Example 3.9.** *Let $\rho$ be the 2-d representation of $G_K$ on $(\mathbb{C}_K)^2$ with matrix given by:*

$$\rho(\sigma) = \begin{pmatrix} 1 & \log\chi(\sigma) \\ 0 & 1 \end{pmatrix}. \tag{32}$$

*It is an extension of $\mathbb{C}_K(0)$ by itself. Now, differentiation of $\rho(\sigma)^t$ as in (30), yields:*

$$\phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{33}$$

*which isn't semisimple, so $\rho$ isn't Hodge-Tate and we see that $\mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K)$ isn't closed under extensions.*

## References