(φ_L, Γ_L)-modules.

Arun Soor

Almost everything below is lifted from [Sch17] so please see there for all the details. All typos and mistakes below are my own. As a disclaimer: I make no claim to understand p-adic Hodge theory.

1 Introduction

One of the goals of number theory is to understand the absolute Galois group of a number field. Since this is extremely difficult we attempt to simplify the problem by working "one place at a time". Let $L/\mathbb{Q}_p$ be a finite extension with ring of integers o and residue field k. By local class field theory we have the local Artin map

$$\mathrm{rec} : L^\times \to G_L^{\mathrm{ab}} \tag{1}$$

characterised by the property that "every uniformizer of L acts by the Frobenius", and for every finite abelian extension L'/L, rec induces an isomorphism $L^\times / \operatorname{Norm}_{L'/L}(L') \xrightarrow{\sim} \operatorname{Gal}(L'/L)$. In fact rec induces an isomorphism from the profinite completion

$$\mathrm{rec} : \widehat{L^\times} \xrightarrow{\sim} G_L^{\mathrm{ab}}. \tag{2}$$

In other words we have a near total understanding of the "1-dimensional" representations of G_L. We would like to understand $\operatorname{Rep}_o(G_L) :=$ the category of finitely generated o-modules equipped with a continuous G_L-action. The paradigm of (φ_L, Γ_L)-modules is to understand this category by replacing the Galois action by a simpler group at the expense of introducing a much larger coefficient ring.

2 Definition of (φ_L, Γ_L)-modules

Let $\pi \in L$ be a uniformizer and set

$$\mathscr{A}_L := \varprojlim_m o(\!(X)\!)/\pi^m = \left\{ \sum_{i\in\mathbb{Z}} a_i X^i : a_i \xrightarrow{i\to-\infty} 0 \right\}. \tag{3}$$

equipped with the Gauss norm/valuation this is a DVR with residue field $k(\!(X)\!)$. The ring $\mathscr{A}_L$ can be viewed naturally as a subset of $o^{\mathbb{Z}}$ and hence acquires a second topology (besides the valuation topology), which is called the weak topology since it is the topology of coefficientwise convergence.

A Frobenius power series is an $\phi(X) \in o[\![X]\!]$ such that $\phi(X) = X^q \mod \pi$ and $\phi(X) = \pi X \mod X^2$. The choice of ϕ yields a Lubin-Tate formal group law (depending only on π up to isomorphism), $F_\phi(X, Y) \in o[\![X, Y]\!]$ such that $\phi \in \operatorname{End}(F_\phi)$. Moreover there is an injective ring homomorphism $[\cdot]_\phi : o \to \operatorname{End}(F_\phi)$ such that $[\pi]_\phi = \phi$. This gives an action of the monoid $o \setminus \{0\}$ on $\mathscr{A}_L$ by $a.f(X) := f([a]_\phi(X))$. Since $o \setminus \{0\} = \pi^{\mathbb{N}_0} o^\times$ this can be viewed as an action by $\Gamma_L := o^\times$ and the endomorphism φ_L sending $f(X) \mapsto f([\pi]_\phi(X))$. These actions are both continuous for the (weak) topology.

Example 2.1. *When $L = \mathbb{Q}_p$ one takes $\pi = p$, $\varphi = (1+X)^p - 1$, then $F_\phi(X,Y) = (X+1)(Y+1) - 1$ is the multplicative law and $[a]_\phi = (1+X)^a - 1$ for $a \in \mathbb{Z}_p$.*

Any finitely generated $\mathscr{A}_L$-module M acquires a canonical topology which is the quotient topology of the weak topology along any surjection $\mathscr{A}_L^{\oplus n} \to M$. The category of (φ_L, Γ_L)-modules is the category of finitely generated $\mathscr{A}_L$-modules M equipped with a semilinear continuous action of Γ_L and a commuting φ_L-linear continuous endomorphism $\varphi_M : M \to M$. A (φ_L, Γ_L)-module M is called *étale* if the map $\varphi_M^{\mathrm{lin}} : \mathscr{A}_L \otimes_{\mathscr{A}_L, \varphi_L} M \to M$ sending $f \otimes m \mapsto f\varphi_M(m)$, is an isomorphism[1]. We will sketch the construction of the explicit equivalence

$$\mathrm{Rep}_o(G_L) \cong \mathrm{Mod}^{\mathrm{et}}(\mathscr{A}_L) := \{\text{category of étale } (\varphi_L, \Gamma_L) - \text{modules}\}. \tag{4}$$

3 A generalisation of the Fontaine-Winterberger theorem

Fix an algebraic closure $\overline{L}$ of L inside $\mathbb{C}_p$. Let $\mathfrak{M} \subset o_{\overline{L}}$ be the maximal ideal and, for each $n \geq 1$ set $\mathscr{F}_n := \ker([\pi^n]_\phi)(\mathfrak{M})$ and $L_n := L(\mathscr{F}_n)$. Set $T := \varprojlim_n \mathscr{F}_n$ and $L_\infty \bigcup_n L_n$. Of course, $\mathrm{Gal}(L_n/L)$ acts on $\mathscr{F}_n$. In fact $\mathscr{F}_n$ turns out to be a free rank 1 $o/\pi^n o$-module and hence T is free of rank 1 as an o-module. Hence, the choice a basis element $t \in T$ (i.e., a compatible system of torsion points), induces the Lubin-Tate character

$$\chi_L : \mathrm{Gal}(L_\infty/L) \to o^\times = \Gamma_L, \tag{5}$$

which turns out to be an isomorphism. The extensions L_n/L are totally ramified, in particular, L_∞ has residue field k.

Example 3.1. *In our running example with $L = \mathbb{Q}_p$, $\pi = p$ and $F_\phi = \widehat{\mathbb{G}}_m$ we obtain $\mathcal{F}_n = \{\zeta - 1 : \zeta^{p^n} = 1\}$ and $L_n = \mathbb{Q}_p(\zeta_{p^n})$, and χ_L is the cyclotomic character.*

Recall that an intermediate field $L \subset K \subset \mathbb{C}_p$ is called *perfectoid* if it is complete, indiscretely valued and $(o_K/po_K)^p = o_K/po_K$. Given such a field we set $o_{K^\flat} := \varprojlim_{x \mapsto x^q} o_K/\pi o_K$. This is a perfect k-algebra. Given a compatible system $(\alpha_i)_i \in o_{K^\flat}$ we can choose arbitary lifts a_i of α_i to o_K and set $\alpha^\sharp := \lim_{i\to\infty} a_i^{q^i}$ to obtain a well-defined element $\alpha^\sharp \in o_K$. This map allows us to define a norm[2] $|\cdot|_{K^\flat}$ on $o_{K^\flat}$ by $|\alpha|_{K^\flat} := |\alpha^\sharp|_K$. With respect to the norm $|\cdot|_{K^\flat}$, $o_{K^\flat}$ has the same valuation monoid as o_K. The maximal ideal of $o_{K^\flat}$ is given in terms of $|\cdot|_{K^\flat}$ in the usual way and it turns out that the residue fields of o_K and $o_{K^\flat}$ are canonically isomorphic. The fraction field $K^\flat$ of $o_{K^\flat}$ together with $|\cdot|_{K^\flat}$ is then a perfect nonarchimedean field of characteristic p.

We have two examples of perfectoid fields, namely $\widehat{L}_\infty$ and $\mathbb{C}_p$. The natural map $o_{L_\infty}/\pi \to o_{\mathbb{C}_p}/\pi$ is injective and hence $\widehat{L}_\infty \hookrightarrow \mathbb{C}_p^\flat$ naturally. The "tilting correspondence" due to Scholze says that $K \mapsto K^\flat$ gives an inclusion-respecting bijection

$$\{\text{perfectoid fields } \widehat{L}_\infty \subset K \subset \mathbb{C}_p\} \leftrightarrow \{\text{complete and perfect fields } \widehat{L}_\infty^\flat \subset F \subset \mathbb{C}_p^\flat\} \tag{6}$$

[1] Setting $Y = \mathrm{Spec}(\mathscr{A}_L)$, we can informally think of this condition as some kind of φ_L-equivariance or descent datum.

[2] The multiplicativity of this map is immediate, and the additivity follows from the formulas:

$$|(\alpha+\beta)^\sharp| = |\lim_{i\to\infty}(a_i+b_i)^{q^i}| = \lim_{i\to\infty}|a_i+b_i|^{q^i} \leq \lim_{i\to\infty}\max(|a_i|,|b_i|)^{q^i}$$
$$= \max(\lim_{i\to\infty}|a_i|^{q^i}, \lim_{i\to\infty}|b_i|^{q^i}) = \max(|\alpha^\sharp|,|\beta^\sharp|).$$

whose inverse is given by *untilting* $F \mapsto F^\sharp$ (we do not have time to discuss this).

The group G_L acts naturally on $o_{\mathbb{C}_p^\flat}$ by $\sigma \cdot (\dots, a_i \bmod \pi, \dots) = (\dots, \sigma(a_i) \bmod \pi, \dots)$. This preserves the norm $|\cdot|_{\mathbb{C}_p^\flat}$ and in fact induces a continuous action of G_L on $\mathbb{C}_p^\flat$. The action of $H_L := \mathrm{Gal}(\overline{\mathbb{Q}}_p/L_\infty) \subseteq G_L$ fixes $\widehat{L}_\infty \subseteq \mathbb{C}_p$ and $\widehat{L}_\infty^\flat \subseteq \mathbb{C}_p^\flat$, by continuity. Hence, we obtain a residual $\Gamma_L = G_L/H_L$-action on $\widehat{L}_\infty^\flat$.

Now let us return to the Tate module T of the Lubin-Tate formal group law F_ϕ. The Frobenius power series property implies that

$$\iota : T \mapsto o_{\widehat{L}_\infty^\flat} \quad (y_n)_{n\geq 1} \mapsto (\dots, y_n \mod \pi o_{\widehat{L}_\infty^\flat}, \dots, y_1 \mod \pi o_{\widehat{L}_\infty^\flat}, 0), \tag{7}$$

is a well-defined map (but not a homomorphism). The image of the basis element gives $\omega := \iota(t) \in o_{\widehat{L}_\infty^\flat}$. By the ramification theory of the Lubin-Tate extensions, it follows that $|\omega|_\flat = |\pi|^{q/(q-1)} < 1$. Hence $X \mapsto \omega$ gives a ring map $k[\![X]\!] \to o_{\widehat{L}_\infty^\flat}$ which extends to $k(\!(X)\!) \hookrightarrow \widehat{L}_\infty^\flat$. We define the *field of norms* $\mathbf{E}_L \cong k(\!(X)\!)$ to be the image of this map. This subfield and the map ι have the following properties:

(i) For any $\gamma \in \Gamma_L$ we have $\gamma(\omega) = \overline{[\chi_L(\gamma)]_\phi}(\omega)$. In particular (by continuity) it follows that the Γ_L-action on $\widehat{L}_\infty^\flat$ preserves $\mathbf{E}_L$.

(ii) $\widehat{\mathbf{E}_L^{\mathrm{perf}}} = \widehat{L}_\infty^\flat$ and $\widehat{\mathbf{E}_L^{\mathrm{sep}}} = \widehat{\overline{\mathbf{E}}_L} = \mathbb{C}_p^\flat$; we say that $\widehat{\mathbf{E}_L^{\mathrm{perf}}}$ (resp. $\mathbf{E}_L^{\mathrm{sep}}$), is a *decompletion* of $\widehat{L}_\infty^\flat$ (resp. $\mathbb{C}_p^\flat$).

In the preceding we introduced the *perfect hull* $\mathbf{E}_L^{\mathrm{perf}} := \{x \in \overline{\mathbf{E}}_L : x^{p^m} \in \mathbf{E}_L \text{ for some } m \geq 0\}$. By general field theory and the above facts, we obtain isomorphisms by restriction

$$\mathrm{Aut}^{\mathrm{cts}}(\mathbb{C}_p^\flat, \widehat{L}_\infty^\flat) \xrightarrow{\sim} \mathrm{Gal}(\overline{\mathbf{E}}_L/\mathbf{E}_L^{\mathrm{perf}}) \xrightarrow{\sim} \mathrm{Gal}(\mathbf{E}_L^{\mathrm{sep}}/\mathbf{E}_L) =: H_{\mathbf{E}_L}; \tag{8}$$

here the first is by continuity and the second is by property of the perfect hull. On the other hand we have by continuity an isomorphism

$$H_L = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\widehat{L}_\infty) \xleftarrow{\sim} \mathrm{Aut}^{\mathrm{cts}}(\mathbb{C}_p, \widehat{L}_\infty), \tag{9}$$

and the untilting-tilting formalism gives a bijection

$$\mathrm{Aut}^{\mathrm{cts}}(\mathbb{C}_p, \widehat{L}_\infty) \to \mathrm{Aut}^{\mathrm{cts}}(\mathbb{C}_p^\flat, \widehat{L}_\infty^\flat), \quad \sigma \mapsto \sigma^\flat, \quad \sigma^\sharp \mapsfrom \sigma, \tag{10}$$

which is in fact an isomorphism of topological groups (this is non-trivial to verify). The composite isomorphism $H_L \xrightarrow{\sim} H_{\mathbf{E}_L}$ is identified with $\sigma \mapsto \sigma^\flat$.

Example 3.2. *In our running example with $L = \mathbb{Q}_p$, $\pi = p$, $F_\phi = \widehat{\mathbb{G}}_m$ one has $L_\infty = \mathbb{Q}_p(\zeta_{p^\infty})$. Fixing a compatible system $(\zeta_{p^n})_n$, we obtain $\omega := (\dots, \zeta_{p^2} - 1 \mod p, \zeta_p - 1 \mod p, 0) \in o_{\widehat{L}_\infty^\flat}$ and $\mathbb{F}_p(\!(X)\!) \xrightarrow{\sim} \mathbf{E}_{\mathbb{Q}_p}$ via $X \mapsto \omega$. Then (ii) above tells us that this gives $\widehat{\mathbb{F}_p(\!(X)\!)(X^{1/p^\infty})} \xrightarrow{\sim} \widehat{L}_\infty^\flat$. Restiction of the tilted action to $\mathbf{E}_{\mathbb{Q}_p}^{\mathrm{sep}}$ gives an isomorphism $G_{\mathbb{Q}_p(\zeta_{p^\infty})} \cong G_{\mathbb{F}_p(\!(X)\!)}$.*

4 The coefficient ring revisited

In the previous section we constructed an embedding $k(\!(X)\!) \hookrightarrow \widehat{L}^{\flat}_{\infty}$ whose image was defined to be $\mathbf{E}_L$. We would now like to lift this to am algebra morphism j:

$$\begin{array}{ccc} \mathscr{A}_L & \xrightarrow{\ \mathrm{j}\ } & W(\mathbf{E}_L)_L \\ {\scriptstyle \bmod \pi}\downarrow & & \downarrow{\scriptstyle \Phi_0} \\ k(\!(X)\!) & \xrightarrow{\ \sim\ } & \mathbf{E}_L \end{array} \tag{11}$$

such that j is equivariant for the Γ_L-actions (the Γ_L-action on $W(\mathbf{E}_L)_L$ being induced by functoriality of the ramified Witt vector construction) and sends the action of φ_L to the Frobenius Fr on $W(\mathbf{E}_L)_L$. Here Φ_0 is the 0^{th} ghost component map. In order to construct such a morphism we need to specify the image of $X \in \mathscr{A}_L$, in other words, we need to lift $\omega \in \mathbf{E}_L$ to an element $\omega_\phi \in W(\mathbf{E}_L)_L$. One would usually use the Teichmuller representative $\tau : \mathbf{E}_L \to W(\mathbf{E}_L)_L$ to achieve this, however, it doesn't have the right equivariance properties, and so it needs to be modified.

Let $\mathbb{M}_{\mathbf{E}_L} := \Phi_0^{-1}(\mathfrak{m}_{\mathbf{E}_L}) \subseteq W(\mathbf{E}_L)_L$; this is a maximal ideal. Via the Lubin-Tate formal group law F_ϕ, $\mathbb{M}_{\mathbf{E}_L}$ acquires the structure of an o-module. It turns out that $[\pi]_\phi \circ \mathrm{Fr}^{-1}$ is a well-defined o-module endomorphism of $\mathbb{M}_{\mathbf{E}_L}$. Ignoring questions of convergence we can define an o-module endomorphism

$$\{\cdot\} : \mathbb{M}_{\mathbf{E}_L} \to \mathbb{M}_{\mathbf{E}_L} \quad \{\alpha\} := \lim_{i \to \infty} ([\pi]_\phi \circ \mathrm{Fr}^{-1})^i(\alpha), \tag{12}$$

the definition of $\{\alpha\}$ is rigged so that $[\pi]_\phi(\{\alpha\}) = \mathrm{Fr}(\{\alpha\})$. Hence, if one defines

$$\iota_\phi := \text{the composite } (T \xrightarrow{\iota} \mathfrak{m}_{\mathbf{E}_L} \xrightarrow{\tau} \mathbb{M}_{\mathbf{E}_L} \xrightarrow{\{\cdot\}} \mathbb{M}_{\mathbf{E}_L}) \tag{13}$$

then one can verify that $\mathrm{Fr}(\iota_\phi(t)) = \iota_\phi(\pi \cdot t)$. It turns out that $\Phi_0 \iota_\phi = \iota$ and ι_ϕ also has the right Γ_L-equivariance.

Therefore we choose $\omega_\phi := \iota_\phi(t)$ and the o-algebra map $\mathrm{j} : \mathscr{A}_L \to W(\mathbf{E}_L)_L$ is detemined by $X \mapsto \omega_\phi$. This is Γ_L-equivariant and satisfies $\mathrm{j} \circ \varphi_L = \mathrm{Fr} \circ \mathrm{j}$. It follows that the image $\mathbf{A}_L := \mathrm{im}(\mathrm{j})$ is equipped with a (φ_L, Γ_L)-action, which coincides with that inherited from the (Fr, Γ_L) action on $W(\mathbf{E}_L)_L$. The map j also turns out to be a topological embedding for the respective weak topologies so that $\mathrm{j} : \mathscr{A}_L \to \mathbf{A}_L$ is a topological isomorphism.

We now "redefine" the category of (φ_L, Γ_L)-modules by replacing instances of $\mathscr{A}_L$ in the previous definition by $\mathbf{A}_L$.

5 The functors

By the previous we have constructed a (Fr, Γ_L)-stable subalgebra $\mathbf{A}_L \subseteq W(\mathbf{E}_L)_L$, which is naturally contained in $W(\mathbf{E}_L^{\mathrm{sep}})_L$. We define $\mathbf{B}_L$ to be the fraction field of $\mathbf{A}_L$: note that the residue field of $\mathbf{B}_L$ is identified with $\mathbf{E}_L$. The next techical input (which we do not have time to prove) is the following:

Proposition 5.1. *There is a unique intermediate ring*

$$\mathbf{A}_L \subseteq \mathbf{A}_L^{\mathrm{nr}} \subseteq W(\mathbf{E}_L^{\mathrm{sep}})_L \tag{14}$$

such that:

- $\mathbf{A}_L^{\mathrm{nr}}$ *is a complete DVR with uniformizer* π*;*
- $\mathbf{B}_L^{\mathrm{nr}} := \mathrm{Frac}(\mathbf{A}_L^{\mathrm{nr}})$ *is the unique subextension of* $\mathrm{Frac}(W(\mathbf{E}_L^{\mathrm{sep}})_L)$ *which is a maximal unramified extension of* $\mathbf{E}_L$*;*
- $\Phi_0 : \mathbf{A}_L^{\mathrm{nr}}/\pi \xrightarrow{\sim} \mathbf{E}_L^{\mathrm{sep}}$ *is an isomorphism;*
- $\mathbf{A}_L^{\mathrm{nr}}$ *is preserved by the Frobenius* Fr *and the* G_L *action inherited from* $W(\mathbf{E}_L^{\mathrm{sep}})_L$ *(the latter coming from tilting equivalence); also* H_L *fixes* $\mathbf{A}_L$*.*

Finally we define

$$\mathbf{A} := \text{ closure of } \mathbf{A}_L^{\mathrm{nr}} \subseteq W(\mathbf{E}_L^{\mathrm{sep}})_L \text{ w.r.t the } \pi-\text{adic topology.} \tag{15}$$

Since the G_L-action on Witt vectors is "coefficientwise", we see that the G_L-action commutes with Fr and $(W(\mathbf{E}_L^{\mathrm{sep}})_L)^{\mathrm{Fr}=1} = W(k)_L = o$. In particular $\mathbf{A}^{\mathrm{Fr}=1} = o$. Hence, we can define, for $M \in \mathrm{Mod}^{\mathrm{et}}(\mathbf{A}_L)$, the o-linear G_L-representation

$$\mathscr{V}(M) := (\mathbf{A} \otimes_{\mathbf{A}_L} M)^{\mathrm{Fr}\otimes\varphi_M=1}, \tag{16}$$

here G_L acts diagonally and through the residual Γ_L-action on M.

On the other hand, by the property of unramified extensions, the G_L-action on $W(\mathbf{E}_L^{\mathrm{sep}})_L$ gives natural isomorphisms

$$H_L \xrightarrow{\sim} \mathrm{Gal}(\mathbf{B}_L^{\mathrm{nr}}/\mathbf{B}_L) \xrightarrow{\sim} \mathrm{Gal}(\mathbf{E}_L^{\mathrm{sep}}/\mathbf{E}_L), \tag{17}$$

so it is not so surprising (though, we do not prove it), that $\mathbf{A}^{H_L} = \mathbf{A}_L$. Given $V \in \mathrm{Rep}_o(G_L)$, the $\mathbf{A}$-module $\mathbf{A} \otimes_o V$ acquires the diagonal G_L-action and the Fr-linear endomorphism $\varphi := \mathrm{Fr} \otimes \mathrm{id}$. Thus the $\mathbf{A}_L$-module

$$\mathscr{D}(V) := (\mathbf{A} \otimes_o V)^{H_L} \tag{18}$$

acquires a residual Γ_L-action and a commuting $\varphi_{\mathscr{D}(V)} := \varphi|\mathscr{D}(V)$-action. The main theorem is

Theorem 5.2 (Fontaine, Kisin-Ren, Colmez, Schneider)**.** *The functors*

$$\mathscr{V} : \mathrm{Mod}^{\mathrm{et}}(\mathbf{A}_L) \leftrightarrows \mathrm{Rep}_o(G_L) : \mathscr{D}, \tag{19}$$

give an equivalence of categories.

Implicit in this is of course the fact that the functors are well-defined, i.e., $\mathscr{V}(M)$ and $\mathscr{D}(V)$ are finitely generated, the actions are continuous and $\mathscr{D}(V)$ is "étale". We give a sketch of the proof in the case of π-torsion coefficients, i.e., the equivalence

$$\mathscr{V} : \mathrm{Mod}^{\mathrm{et}}(\mathbf{E}_L) \leftrightarrows \mathrm{Rep}_k(G_L) : \mathscr{D}, \tag{20}$$

given by $\mathscr{V}(M) := (\mathbf{E}_L^{\mathrm{sep}} \otimes_{\mathbf{E}_L} M)^{\varphi=1}$ and $\mathscr{D}(V) := (\mathbf{E}_L^{\mathrm{sep}} \otimes_k V)^{H_L}$. For the general case one can use a *dévissage* argument to bootstrap this to π^m-torsion coefficients and then take limits.

By an argument involving Hilbert 90 the $\mathbf{E}_L^{\mathrm{sep}}$-vector space $\mathbf{E}_L^{\mathrm{sep}} \otimes_k V$ has a basis by H_L-fixed vectors. Using this basis it is easily verified that $\mathscr{D}(V)$ is finitely generated and the natural morphism

$$\mathbf{E}_L^{\mathrm{sep}} \otimes_{\mathbf{E}_L} \mathscr{D}(V) \xrightarrow{\sim} \mathbf{E}_L^{\mathrm{sep}} \otimes_k V \tag{21}$$

is an isomorphism (one says that V is *admissible*). Hence using (21) we calculate

$$\mathscr{V}(\mathscr{D}(V)) = (\mathbf{E}_L \otimes_{\mathbf{E}_L} \mathscr{D}(V))^{\varphi=1} \xrightarrow{\sim} (\mathbf{E}_L^{\mathrm{sep}} \otimes_k V)^{\varphi=1} = (\mathbf{E}_L^{\mathrm{sep}})^{\mathrm{Fr}=1} \otimes_k V = V. \tag{22}$$

On the other hand, for $M \in \mathrm{Mod}^{\mathrm{et}}(\mathbf{E}_L)$ it is a consequence of Galois/étale descent (here is where we use that φ_M^{lin} is an isomorphism), that

$$\dim_k \mathscr{V}(M)^{\varphi=1} = \dim_{\mathbf{E}_L^{\mathrm{sep}}} \mathbf{E}_L^{\mathrm{sep}} \otimes_{\mathbf{E}_L} M = \dim_{\mathbf{E}_L} M \tag{23}$$

and the natural map

$$\mathbf{E}_L^{\mathrm{sep}} \otimes_k \mathscr{V}(M) \xrightarrow{\sim} \mathbf{E}_L^{\mathrm{sep}} \otimes_{\mathbf{E}_L} M, \tag{24}$$

is an isomorphism. Hence using (24) we calculate

$$\mathscr{D}(\mathscr{V}(M)) = (\mathbf{E}_L^{\mathrm{sep}} \otimes_k \mathscr{V}(M))^{H_L} \xrightarrow{\sim} (\mathbf{E}_L^{\mathrm{sep}} \otimes_{\mathbf{E}_L} M)^{H_L} = (\mathbf{E}_L^{\mathrm{sep}})^{H_L} \otimes_{\mathbf{E}_L} M = M. \tag{25}$$

which completes our proof sketch.

References

[Sch17] Peter Schneider. *Galois representations and* (φ, Γ)*-modules*. Vol. 164. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2017, pp. vii+148. ISBN: 978-1-107-18858-7. DOI: 10.1017/9781316981252. URL: https://doi.org/10.1017/9781316981252.